Pod Meeting 7

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1 Background

This handout is intended to be an introductory document to AMC 10-12 level combinatorics. Counting and probability are topics that are very frequently covered by the AMC 10, and these types of problems are very easy to mess up (i.e. get an off-by-one error) if you're not careful and consistent with your counting strategies. Typically, I find combinatorics and geometry to be the hardest types of problems to master at the AMC level. There are a wide variety of techniques and approaches to solve combo problems, but determining which approaches to use is often the most important step.

2 Fundamentals

- **Product Rule**

  Suppose we have some procedure that can be broken down into k tasks (in a specific order). If there are n_1 ways to do the first task, n_2 ways to do the second task, and so on until there are n_k ways to do the k^{th} task, then the total number of ways to complete the procedure is $n_1 \times n_2 \times \cdots \times n_k$.

- **Permutations**

  A permutation is an arrangement of items where order matters.

  Definition: The total number of permutations of k elements taken from a set of n elements (without repetition) is commonly denoted ${}_nP_k$:

  $${}_nP_k = n(n-1)(n-2)\cdots(n-k+1) = \frac{n!}{(n-k)!}$$

  where $n! = 1 \times 2 \times \cdots \times n$ is the factorial of n.

  To see why this formula is true, we simply use the product rule above in each stage of selection (we start with n elements, and slowly select them until only $n-k$ remain)

  For example, if we have 10 people and wish to pick 5 of them and line them up in a specific order, then there are ${}_{10}P_5 = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 = 30240$ ways to do so.

- **Combinations**

  A combination is an arrangement of items where order does not matter.

> **Definition**: The total number of combinations of k elements taken from a set of n elements (without repetition) is commonly denoted ${}_nC_k$. In fact, combinations are much more likely to come up in contests that we have a special notation for them: $\binom{n}{k}$.
>
> $${}_nC_k = \binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} = \frac{n!}{k!(n-k)!}$$

To explain why this formula is true, we note that there are exactly $k!$ ways to rearrange k distinct objects. Since order mattered in our counting of permutations, we must divide by $k!$ to get our value for the number of combinations.

For example, if we had 10 people and we wanted to choose a set of 5 of them where order **doesn't matter**, then there are $\binom{10}{5} = 252$ ways to do so.

3 Complementary Counting

Before we get to the rest of our lesson, let me introduce a couple of key types of counting problems and a couple of general approaches. The first one is complementary counting. This technique may be hard to spot sometimes, but it can be very effective in problems where there seems to be no easy approach. Essentially, this involves counting the opposite of what we originally wanted, then subtract that from the total number of cases.

Example 1 (Complementary Counting)

How many ways are there to arrange 5 Ts and 5 Ss in a row, such that there exists at least one pair of two letters that are both Ts?

For example, TSTSTTSSST works since the 5^{th} and 6^{th} letters are both Ts.

Example 2 (2006 AMC 10A Problem 21)

How many four-digit positive integers have at least one digit that is a 2 or a 3?

4 Casework

One of the main things about combo is that there isn't really much theory to learn. As a result, getting really good at combo really just relies on getting as much practice as possible. Often, breaking down a combo problem into many small chunks is very helpful in the solving process, but always make sure that you're not forgetting anything.

Example 3 (Casework)

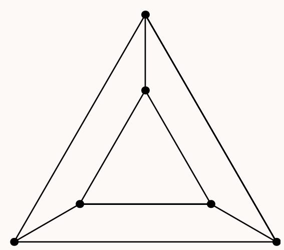
How many squares of integer side length can be formed on a 5 by 5 grid of dots?

Example 4 (2013 AMC 12A Problem 15)

Rabbits Peter and Pauline have three offspring - Flopsie, Mopsie, and Cotton-tail. These five rabbits are to be distributed to four different pet stores so that no store gets both a parent and a child. It is not required that every store gets a rabbit. In how many different ways can this be done?

Example 5 (Mandelbrot 2013-2014)

In the diagram below, how many ways are there to colour two of the dots red, two of the dots blue, and two of the dots green so that dots of the same colour are joined by a segment?



5 Principle of Inclusion-Exclusion

Theorem: Principle of Inclusion Exclusion (2 variables)

$$|A \cup B| = |A| + |B| - |A \cap B|$$

This is what we normally associate with the "Venn Diagram" problems, like the one below:

Example 6 (PIE)

There are three kinds of sciences available for the students to choose - bio, chem, and physics. 50 choose bio in total, 30 choose chem, and 20 choose physics. In addition, 10 people choose both chem and physics, 20 people choose both bio and chem, and 15 people choose bio and physics. Finally, 5 people choose all three. How many students chose at least one science?

An important note to remember when using PIE in contests is recognizing how to strategically overcount/undercount. Typically, memorizing formulas in combo is a bad idea and can easily lead you to a wrong answer.

6 Miscellaneous Tips and Tricks

Theorem: **Stars and Bars 1:**

The number of ways to place n indistinguishable balls into k labelled urns is

$$\binom{n+k-1}{n} = \binom{n+k-1}{k-1}$$

Theorem: **Stars and Bars 2:**

The number of solutions in nonnegative integers to the equation $x_1 + x_2 + \cdots + x_k = n$ is

$$\binom{n+k-1}{n} = \binom{n+k-1}{k-1}$$

Do you see why these two problems are equivalent?

Example 7 (Stars and Bars)

How many ways are there to choose a 5-letter word from the 26-letter English alphabet with replacement, where words that are anagrams are considered the same?

Example 8 (COMC 2018 Problem B4)

Determine the number of 5-tuples of integers $(x_1, x_2, x_3, x_4, x_5)$ such that

(a) $x_i \geq i$ for $1 \leq i \leq 5$

(b) $\displaystyle\sum_{i=1}^{5} x_i = 25$

Here are some more techniques for counting problems:

- **Grouping:** if some elements have to be next to each other, combine them together to be one item.
- **Spacing:** if some elements have to be separated, arrange other elements first, then insert the ones with conditions into the gaps.
- **Special Elements / Positions:** If some of the elements or positions have specific requirements, handle those first
- **Overcounting:** Sometimes it is easier to ignore conditions and overcount, then divide to get the answer (i.e. in some circular arrangement problems)

7 Problems

Of course, the most important part of combo (or any part of math, for that matter) is PRACTICE:

Problem 1 (2017 AMC 10A Problem 8)

At a gathering of 30 people, there are 20 people who all know each other and 10 people who know no one. People who know each other hug, and people who do not know each other shake hands. How many handshakes occur within the group?

Problem 2 (2002 AMC 10B Problem 18)

Four distinct circles are drawn in a plane. What is the maximum number of points where at least two of the circles intersect?

Problem 3 (2017 AMC 10B Problem 13)

There are 20 students participating in an after-school program offering classes in yoga, bridge, and painting. Each student must take at least one of these three classes, but may take two or all three. There are 10 students taking yoga, 13 taking bridge, and 9 taking painting. There are 9 students taking at least two classes. How many students are taking all three classes?

Problem 4 (2003 AIME II Problem 3)

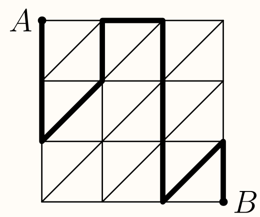
Define a *good word* as a sequence of letters that consists only of the letters A, B, and C - some of these letters may not appear in the sequence - and in which A is never immediately followed by B, B is never immediately followed by C, and C is never immediately followed by A. How many seven-letter good words are there?

Problem 5 (2018 AMC 10A Problem 20)

A scanning code consists of a 7×7 grid of squares, with some of its squares colored black and the rest colored white. There must be at least one square of each color in this grid of 49 squares. A scanning code is called *symmetric* if its look does not change when the entire square is rotated by a multiple of $90°$ counterclockwise around its center, nor when it is reflected across a line joining opposite corners or a line joining midpoints of opposite sides. What is the total number of possible symmetric scanning codes?

Problem 6 (Mandelbrot 2013-2014)

How many paths are there from A to B through the network shown below if you may only move up, down, right, and up-right? A path may not traverse any portion of the network more than once. A sample path is highlighted.



Problem 7 (1998 AIME Problem 7)

Let n be the number of ordered quadruples (x_1, x_2, x_3, x_4) of positive odd integers that satisfy $\sum_{i=1}^{4} x_i = 98$. Find $\frac{n}{100}$.

Problem 8 (2010 AMC 10B Problem 23)

The entries in a 3×3 array include all the digits from 1 through 9, arranged so that the entries in every row and column are in increasing order. How many such arrays are there?

Problem 10

How many ways are there to distribute 11 lollipops to 5 unruly students if every one of them must receive at least one lollipop and exactly one of them insists on getting an even number of lollipops.