# The Quadratic Formula

Quadratic functions and equations

Algebra 1

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**Abstract**

This handout is a transcription and synthesis of the content covered in the Quadratic Functions and Equations unit of the Algebra 1 course from KhanAcademy. In this handout, we will be covering the quadratic formula, the discriminant, and how to use it to solve quadratic equations. Content and questions are courtesy of Sal Khan and KhanAcademy. The images are not mine; they are the property of their respective owners.

## 1 The Quadratic Formula

By now, you should be familiar with the standard form of a quadratic equation. This happens when a quadratic equation is presented in the form

$$ax^2 + bx + c = 0$$

So far, we've covered two methods to solve this equation: factoring and completing the square. However, those two methods only work in certain situations, so we're going to introduce a method that can get the solutions of any quadratic equation simply by plugging in it's coefficients. To do this, we need to isolate the $x$ value algebraically, and we're going to do it by completing the square.

Firstly, divide everything by $a$. Note that 0 divided by any non-zero value is 0, so the right side doesn't change.

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Move the $\frac{c}{a}$ term to the other side, such that both terms on the left side of the equation contain $x$

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

Now, we begin completing the square. To recap, the expansion of the perfect square equation is

$$(a + b)^2 = a^2 + 2ab + b^2$$

We want to rewrite the left side of the equation as the square of a binomial, or some $(x + b)^2$. Observing the $\frac{b}{a}x$ tells us that we need to take half of $\frac{b}{a}$, giving us

$$\frac{b}{a} \times \frac{1}{2} = \frac{b}{2a}$$

Now, we add the square of this value to both sides

$$x^2 + \frac{b}{a}x + (\frac{b}{2a})^2 = -\frac{c}{a} + (\frac{b}{2a})^2$$

Using the perfect square formula, we factor the left side

$$(x + \frac{b}{2a})^2 = -\frac{c}{a} + (\frac{b}{2a})^2$$

Expanding the $(\frac{b}{2a})^2$ gives us

$$(x + \frac{b}{2a})^2 = -\frac{c}{a} + \frac{b^2}{4a^2}$$

Using the common denominator of $4a^2$ gives us

$$(x + \frac{b}{2a})^2 = -\frac{4ac}{4a^2} + \frac{b^2}{4a^2}$$

$$(x + \frac{b}{2a})^2 = \pm\frac{b^2 - 4ac}{4a^2}$$

Take the square root of both sides

$$\sqrt{(x + \frac{b}{2a})^2} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$

Subtract $\frac{b}{2a}$ form both sides

$$x = \frac{\pm\sqrt{b^2 - 4ac}}{2a} - \frac{b}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

**Idea:** Given an equation $ax^2 + bx + c = 0$, the roots to this equation are:
$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Let's test this formula on the equation $x^2 + 4x - 21 = 0$. In this case, we have $a = 1$, $b = 4$, and $c = -21$. Plugging these values into the formula give us

$$x = \frac{-4 \pm \sqrt{(-4)^2 - 4(1)(-21)}}{2(1)}$$

$$x = \frac{-4 \pm \sqrt{16 + 84}}{2}$$

$$x = \frac{-4 \pm \sqrt{100}}{2}$$

$$x = \frac{-4 \pm 10}{2}$$

Thus our two solutions are $x = \frac{-4+10}{2} = 3$, $x = \frac{-4-10}{2} = -7$. If we wanted to check these answers, we could also factor the original equation to

$$(x + 7)(x - 3) = 0$$

In which case we also get $x = -7$ or $x = 3$.

Let's try another example, this time the quadratic of $-3x^2 + 12x = -1$. Firstly, move everything to one side

$$-3x^2 + 12x + 1 = 0$$

Observing the equation gives us that $a = -3$, $b = 12$, $c = 1$. Plugging these values into the equation gives us

$$x = \frac{-12 \pm \sqrt{(12)^2 - 4(-3)(1)}}{2(-3)}$$

$$x = \frac{-12 \pm \sqrt{144 + 12}}{-6}$$

$$x = \frac{-12 \pm \sqrt{156}}{-6}$$

$$x = \frac{-12 \pm 2\sqrt{39}}{-6}$$

$$x = -\frac{-6 \pm \sqrt{39}}{3}$$

$$x = 2 \pm \frac{\sqrt{39}}{3}$$

Making our solutions $x = 2 + \frac{\sqrt{39}}{3}$ and $x = 2 - \frac{\sqrt{39}}{3}$

### 1.1 Questions

Solve for x:
1: $x^2 + 6x - 14 = 0$
2: $8x^2 + 14x - 15 = 0$
3: $2x^2 - 7x = 5$
4: $3x^2 = 10x + 4$
5: $4x^2 - 6x = 14$
6: $x^2 + 8x - 4 = 0$
Extra Practice: (Courtesy of KhanAcademy) Quadratic Formula

## 2 The Discriminant (nature of roots)

Final example, this time with the equation

$$3x^2 + 6x + 10 = 0$$

Observing the equation, we get $a = 3$, $b = 6$, and $c = 10$. Plugging these values into the equation give us

$$x = \frac{-6 \pm \sqrt{(6)^2 - 4(3)(10)}}{2(3)}$$

$$x = \frac{-6 \pm \sqrt{36 - 120}}{6}$$

$$x = \frac{-6 \pm \sqrt{-84}}{6}$$

However, we've run into a problem. Observing $\sqrt{-84}$ tells us that our equation actually has no solutions, because the square of a real number can never be negative. So, how are we able to tell when a quadratic will have no solutions?

Look at the original quadratic equation, especially the part under the square root symbol ($b^2 - 4ac$). Mathematicians call this part the discriminant, or represent it using the $\Delta$ symbol. This discriminant has several properties. As we've seen, when the discriminant is negative, the equation will have no solutions. Let's introduce a few more properties of the discriminant that you can use to predict the nature of the roots of any quadratic equation:

**The Nature of Roots:**

Given an equation $ax^2 + bx + c = 0$, the discriminant of the equation ($\Delta$) represents $b^2 - 4ac$

When $\Delta$ is a perfect square, the equation is always factorable and has two distinct, rational roots.

When $\Delta$ is positive but not a perfect square, the equation has two distinct, irrational roots.

When $\Delta = 0$, the equation has a single rational root.

When $\Delta < 0$, the equation has no real roots, but two distinct, imaginary roots.

## 2.1 Questions

1. Calculate the discriminant of each and use it to determine the nature of the roots:
(a) $3x^2 - 2x = 5$
(b) $4x(3 - x) = 9$
(c) $x^2 - \sqrt{3} + 7 = 0$

2. Find all real values of p for which
(a) $2x^2 – px + 3p = 0$ has equal roots
(b) $x^2 + px + p^2 = 0$ has real and distinct roots

Extra Practice: (Courtesy of KhanAcademy) Number of solutions to quadratic equations

# 3 Answer Key

**1.1 The Quadratic Formula:**
1: $x = -3 \pm \sqrt{23}$
2: $x = \frac{3}{4}$ or $-\frac{5}{2}$
3: $x = \frac{7 \pm \sqrt{89}}{4}$
4: $x = \frac{5 \pm \sqrt{37}}{3}$
5: $x = \frac{3 \pm \sqrt{65}}{8}$
6: $x = -4 \pm 2\sqrt{5}$

**2.1 The Nature of Roots**
1.
(a) Two distinct, rational roots
(b) One rational root/Two equal rational roots
(c) Two distinct, imaginary roots
2.
(a) $p = 0$ or $p = 24$
(b) No solutions