Completing the Square

Quadratic Functions and Equations

Algebra 1

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Abstract

This handout is a transcription and synthesis of the content covered in the Quadratic functions and equations of the Algebra 1 course from KhanAcademy. In this handout, we will be covering completing the square. Content and questions are courtesy of Sal Khan and KhanAcademy. The images are not mine; they are the property of their respective owners.

1 Perfect square formula

Let's start by introducing something called the perfect square formula:

Idea: $(a+b)^2 = a^2 + 2ab + b^2$

Proof:

$$\begin{aligned}(a+b)^2 &= (a+b)(a+b) \\ &= a^2 + ab + ba + b^2 \\ &= a^2 + 2ab + b^2\end{aligned}$$

Note that this also works if b is negative:

$$(a-b)^2 = a^2 - 2ab + b^2$$

A good way to remember this formula is that the sign of the $2ab$ corresponds to the sign of the binomial being squared. For example, $(a+b)^2$ has a middle term of $+2ab$, while $(a-b)^2$ has a middle term of $-2ab$. The last term (b^2) is always positive, because the square of any non-complex number can never be negative. This formula comes in useful when factoring quadratics or completing the square (as we will cover later), but for now, try these practice problems:

1.1 Questions

1. Expand $(x-h)^2$
2. Factor $5c^2 - 50c + 125$
3. Expand $(2d+3)^2$
4. Factor $16p^2 - 40pq + 25q^2$

2 Completing the Square

Take the equation $x^2 - 4x - 5 = 0$. We could solve it by factoring, but we're going to introduce a new factoring method called completing the square. First, we move the -5 to the other side by adding 5 to both sides:

$$x^2 - 4x = 5$$

We want the $x^2 - 4x$ side to look like $a^2 - 2ab + b^2$ so we can factor it using the perfect square formula. Observe the terms of the equation:

$$\begin{gathered} x^2 \text{ is } a^2\text{, meaning that this should factor into some } (x-b)^2 \\ -4x \text{ is the } -2ab\text{, and since } a = x\text{, we can solve for our } b \text{ value:} \\ -4x = -2xb \\ b = 2 \end{gathered}$$

Now, what does this "b" really represent? b is a variable that we made up in order to factor the equation, but we can't just magically add it to both sides of the equation. In order to maintain the correctness of our original equation, we add our newly obtained b value of 4 to both sides of the equation, as such

$$x^2 - 4x + 4 = 5 + 4$$

$$x^2 - 4x + 4 = 9$$

Finally, we are able to factor the equation. Observing the three terms gives us that $a = x$ and $b = -2$.

$$(x-2)^2 = 9$$

Bringing everything back to one side gives us

$$(x-2)^2 - 9 = 0$$

which is now in vertex form, thanks to completing the square.
Not all quadratics are this easy to factor however, for example quadratics with a leading coefficient > 1 require a slightly different method:
For example, let's try factoring $10x^2 - 30x - 8 = 0$ by completing the square. Firstly, note that everything is divisible by 2. $\frac{0}{2} = 0$, so we are able to simplify this to

$$5x^2 - 15x - 4 = 0$$

If we want to complete the square, we need to make sure the leading coefficient is 1. To do this, we divide everything by 5

$$x^2 - 3x - \frac{4}{5} = 0$$

Move the $\frac{4}{5}$ to the other side

$$x^2 - 3x = \frac{4}{5}$$

Now we need to complete the square. Observing our terms again gives that our a value is s, and our $-2ab$ value is $-3x$. Dividing both sides by $-2x$ gives us that our b value is $\frac{3}{2}$, thus we want the left side to factor to $(x - \frac{3}{2})^2$. Expanding this gives us $x^2 - 3x + \frac{9}{4}$. Adding $\frac{9}{4}$ to both sides gives us

$$x^2 - 3x + \frac{9}{4} = \frac{4}{5} + \frac{9}{4}$$

$$(x - \frac{3}{2})^2 = \frac{61}{20}$$

If we wanted to, we could simply solve the quadratic by taking the square root.

$$x - \frac{3}{2} = \pm\sqrt{\frac{61}{20}}$$

$$x = \pm\sqrt{\frac{61}{20}} - \frac{3}{2}$$

$$(x - \frac{3}{2})^2 - \frac{61}{20} = 0$$

which are our two solutions. However, completing the square is actually better applied when graphing quadratics, because it helps convert equations to **vertex form**. As a refresher, the vertex form of a quadratic is $y = a(x-h)^2 + k$, which looks very similar to our result of completing the square.

Let's do one more example, but this time in standard form. To graph the equation $3x^2 - 24x + 6$, we can convert it to vertex form by completing the square. First, we can factor out 3

$$y = 3(x^2 - 8x + 2)$$

For convenience, we're actually going to divide both sides by 3 to make completing the square easier

$$\frac{y}{3} = x^2 - 8x + 2$$

Observing the variables tells us that our b value is -4, meaning that we need to add a 16 to the end

$$\frac{y}{3} + 16 = x^2 - 8x + 16 + 2$$

Now, we factor with the perfect square formula, which gives us

$$\frac{y}{3} + 16 = (x-4)^2 + 2$$

$$\frac{y}{3} = (x-4)^2 - 14$$

$$y = 3(x-4)^2 - 42$$

Thanks to completing the square, we've converted the equation to vertex form. The parabola is upwards-opening and has a vertex at (4, -42).

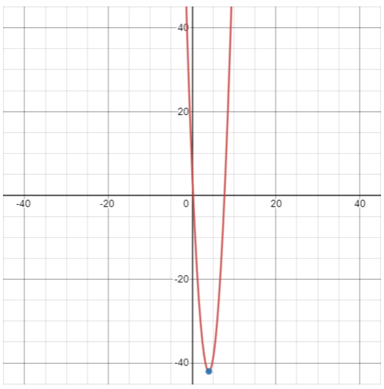

Figure 1: The graph of $y = 3(x-4)^2 - 42$

2.1 Questions

1. Solve by completing the square: $x^2 – 8x + 5 = 0$
2. Solve by completing the square: $x^2 + 12x + 4 = 0$
3. Solve by completing the square: $3x^2 - 12x + 7 = 0$
4. Solve by completing the square: $–2x^2 - 12x - 9 = 0$
5. Solve by completing the square: $4x^2 + 8x - 9 = 0$
6. Solve by completing the square: $-3x^2 - 18x - 35 = 0$

(Courtesy of KhanAcademy): Completing the square (intro).
(Courtesy of KhanAcademy): Completing the square (intermediate).
(Courtesy of KhanAcademy): Solve equations by completing the square.
(Courtesy of KhanAcademy): Completing the square.

3 Answer Key

Perfect Square Formula:
1. $x^2 - 2xh - h^2$
2. $5(x - 5)^2$
3. $4d^2 + 12d + 9$
4. $(4p + 5q)^2$

Completing the Square:
1. $4 \pm \sqrt{11}$
2. $-6 \pm 4\sqrt{2}$
3. $2 \pm \frac{\sqrt{57}}{3}$
4. $-3 \pm \frac{3\sqrt{2}}{2}$
5. $-1 \pm \frac{\sqrt{13}}{2}$
6. No real solution, complex solution of $-3 \pm \frac{2i\sqrt{6}}{3}$