Factoring Quadratics

Quadratic Functions and Equations

Algebra 1

Julian Zhang

April 2021

Abstract

This handout is a transcription and synthesis of the content covered in the Quadratic Functions and Equations unit of Algebra 1 from KhanAcademy. In this handout, we will cover how to solve quadratics by factoring. Content and questions are courtesy of Sal Khan and KhanAcademy. The images are not mine; they are the property of their respective owners.

1 Simple Quadratics

1.1 Factoring by Grouping

Say you wanted to solve the quadratic

$$x^2 + 3x + 4x + 12 = 0$$

Intuitively, you would think to combine like terms, then solve it like a linear equation by isolating the x variable. However, there is an easier to solve quadratics like these (especially when the leading coefficient is 1). When we factor a quadratic equation, we want to write it as the product of two linear equations, and solve each one individually to find the "roots" or "zeroes" of the equation. In this case, we would factor by grouping: Note that the first two terms $(x^2 + 3x)$ have a common factor of x. Let's first factor that out:

$$x(x + 3) + 4x + 12 = 0$$

Now, notice that the last two terms $(4x + 12)$ have a common factor of 4. Factoring that out, we get:

$$x(x + 3) + 4(x + 3) = 0$$

Now, we have only two terms, both containing a $x + 3$. Factoring that out, we finally get

$$(x + 4)(x + 3) = 0$$

To solve, recall the Null Factor Law/Zero Product Property that we learned before. In short, the Null Factor Law states:

For any two integers a and b who's product is 0, either a, b, or both a and b are equal to zero.

Applying that to our now factored quadratic, we get two possible cases. Either $x + 4$ is equal to 0, in which case x is equal to -4, or $x + 3$ is equal to 0, in which case x is equal to -3, and those are our two solutions.

1.2 Questions

1. Solve for x: $x^2 + 2x + 5x + 10 = 0$.
2. Solve for x: $x^2 + 6x - 3x - 18 = 0$.
3. For which values of x does the function $f(x) = x^2 - 8x + 3x - 24$ equal 0?
4. For which values of x does the function $f(x) = 3x^2 + 6x - 4x - 8$ equal 0?

1.3 Simple Trinomials

Sadly, not all quadratics are not presented in easily factorable form. Take for example the quadratic

$$x^2 - 2x - 35 = 0$$

We want to expand the middle term of this quadratic $(-2x)$ into two terms, which can then be easily factored by grouping. To do this, we observe the two coefficients -2 and -35. In order to decompose or "break apart" the middle term, we need two numbers who's sum is -2, and who's product is -35. In other words, find two numbers a and b such that:

$$a + b = -2$$

$$a \times b = -35$$

The reason for this is because when the equation is expanded, the sum of the two x terms have to match $-2x$, and the product of the two numerical teams have to match 35.

After a bit of trial and error, you discover the two numbers a and b are 5 and -7. We can now decompose the original equation into:

$$x^2 + 5x - 7x - 35 = 0$$

The first two terms have a common factor of x:

$$x(x + 5) - 7x - 35 = 0$$

The last two terms terms have a common factor of -7. **Note that you could also say that they have a common factor of 7, but factoring out a 7 would give $(-x - 5)$, which doesn't exactly match $(x + 5)$ and makes it harder to factor**.

$$x(x + 5) - 7(x + 5) = 0$$

Factor out the $x + 5$:

$$(x + 5)(x - 7) = 0$$

Finally, applying the Null Factor Law gives that $x = -5$, or $x = 7$, which are our answers.
Although you could factor every quadratic by taking apart the middle term, there is a very simple trick. We could have skipped the decomposition and grouping steps by just writing $(x+5)(x-7)$ as soon as we got the numbers 5 and -7. To generalize, let's say for any quadratic $x^2 + ax + b = 0$, you want two numbers p and q such that:

$$p + q = a$$

$$p \times q = b$$

Then, we can decompose the middle term into:

$$x^2 + px + qx + b$$

Notice now that q is just $\frac{b}{p}$. We can factor by grouping to get:

$$x(x+p)+q(x+p)$$

$$(x+p)(x+q)=0$$

Finally, to take a and b out of the picture, let's say that any quadratic that can be written as $x^2+(p+q)x+(p\times q)$ can be factored into $(x+p)(x+q)$. Furthermore, if that equation equals zero, our solutions, or "roots" are $-p$ and $-q$.

1.4 Questions

1. Solve for x: $x^2+7x+10=0$
2. Solve for x: $x^2+9x+14=0$
3. Solve for x: $x^2-5x-36=0$
4. Solve for x: $x^2-8x-16=0$
5. For which values of x does the function $f(x)=x^2+2x-15$ equal 0?
6. For which values of x does the function $f(x)=x^2-14x+33$ equal 0?

(Courtesy of KhanAcademy): Quadratics by factoring (intro).

2 Complex Trinomials

Sadly, not all quadratics have a leading coefficient of 1 either. There can be quadratics with other leading coefficients, and they are slightly harder to factor. Firstly, notice that all complex trinomials fall into two categories:

2.1 Leading coefficient can be factored

Look at the trinomial $-2x^2+40x-200=0$. Firstly, note that the leading coefficient is negative, and should be factored out. Additionally, all three terms share a factor of 2, which should also be factored out.

$$-2(x^2-20x+100)=0$$

Now, we use our factoring trick for simple trinomials. We need two numbers that sum to -20, and multiply to 100. These two numbers are -10 and -10. Thus, we can factor it to

$$-2(x-10)(x-10)=0$$

We can apply the Null Factor Law immediately, but also notice that we can divide both sides by -2. 0 divided by anything is 0, so the equation simplifies to

$$(x-10)(x-10)=0$$

Finally, we can apply the Null Factor Law, giving us $x=10$ (one solution).

2.2 Questions

(Courtesy of KhanAcademy): Quadratics by Factoring.

2.3 Leading coefficient can't be factored

Observe the trinomial $2x^2 + 5x + 3 = 0$. In this case, dividing everything by 2 doesn't do us any good because we end up with fractions, and it's much hard to find numbers that add and multiply to a fraction. Instead, we need to decompose the middle term again.
Firstly, we multiply the first and third terms, giving us $2x^2 \times 3 = 6x^2$. Now, we need to find two numbers which sum to $5x$, and multiply to $6x^2$. Those numbers are $2x$ and $3x$, so we can decompose the middle term to get:

$$2x^2 + 2x + 3x + 3 = 0$$

Factor by grouping as before:

$$2x(x+1) + 3(x+1)$$

$$(2x+3)(x+1) = 0$$

Apply the Null Factor Law:

$$\textbf{Case 1: } 2x + 3 = 0$$

$$2x = -3$$

$$x = -\frac{3}{2}$$

$$\textbf{Case 2: } x + 1 = 0$$

$$x = -1$$

This method of factoring is also called decomposition, and while it is similar to the one used for simple trinomials and easy to understand, the is also another method called the "criss cross method". This is the method that most mathematicians usually use, due to being significantly quicker.

2.4 Questions

Factor, then solve:

1. $7m^2 + 6m - 1 = 0$
2. $3k^2 - 10k + 7 = 0$
3. $5x^2 - 36x - 81 = 0$
4. $2x^2 - 9x - 81 = 0$
5. $3n^2 - 16n + 20 = 0$
6. $2r^2 + 7r - 30 = 0$
7. $5k^2 + 8k + 80 = 0$
8. $5x^2 - 14x + 5 = -3$
9. $7p^2 - 20p = -12$
10. $3v^2 - 14v + 11 = 60$

3 Answer Key:

Factoring by Grouping:

1. -5 or -2
2. 6 or -3
3. -3 or 8
4. -2 or $\frac{4}{3}$

Simple Trinomials:

1. -4 or -3
2. -7 or -2
3. -4 or 9
4. 4
5. -5 or 3
6. 3 or 11

Complex Trinomials:

1. $\frac{1}{7}$ or -1
2. $\frac{7}{3}$ or 1
3. $-\frac{9}{5}$ or 9
4. $-\frac{9}{2}$ or 9
5. $\frac{10}{3}$ or 2
6. $\frac{5}{2}$ or -6
7. Not factorable
8. $\frac{5}{4}$ or 2
9. $\frac{6}{7}$ or 2
10. $\frac{7}{3}$ or -7