Vertex Form

Quadratic Functions and Equations
Algebra 1

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Abstract

This handout is a transcription and synthesis of the content covered in the Quadratic Functions and Equations unit of Algebra 1 from KhanAcademy. In this handout, we will be solving quadratics by taking the square root, and how to graph them using vertex form. Content and questions are courtesy of Sal Khan and KhanAcademy. The images are not mine; they are the property of their respective owners.

1 Solving with the square root

Solve $2x^2 + 3 = 75$.

With quadratics like this one, we can solve it simply by treating it as if it were a linear equation, and isolating the variable x.
Subtract 3 from both sides:

$$2x^2 = 72$$

Divide both sides by 2:

$$x^2 = 36$$

Now something important to note here is that the answer is not just 6. $\sqrt{36}$ asks for a number which, when multiplied by itself gives 36. This number could not only be 6, but also -6, because the product of two negatives equals a positive. Thus, we get the solutions of

$$x = \pm 6$$

The plus-minus ($\pm$) symbol denotes that the solution could either be positive or negative 6, something that will be very useful in the future - but for now, let's try another example, this time in factored form.

Solve $(x + 3)^2 - 4 = 0$

Add 4 to both sides:

$$(x + 3)^2 = 4$$

Take the square root of both sides:

$$(x + 3) = \pm 2$$

If $x + 3 = 2$, $x = -1$.
If $x + 3 = -2$, $x = -5$.
Thus, $x = -1$ or -5.

One last example, this time illustrating how taking the square root can help with graphing a quadratic:
For which x values does the function $f(x) = (x-2)^2 - 9$ intersect the x-axis?

An equation only intersects the x-axis when its y value is 0. To solve, we sub in $f(x) = 0$.

$$(x-2)^2 - 9 = 0$$

Add 9 to both sides:

$$(x-2)^2 = 9$$

Take the square root of both sides:

$$x - 2 = \pm 3$$

If $x - 2 = 3$, $x = 5$.
If $x - 2 = -3$, $x = -1$.
Thus, f(x) intercepts the x-axis at points $(5, 0)$, $(-1, 0)$.

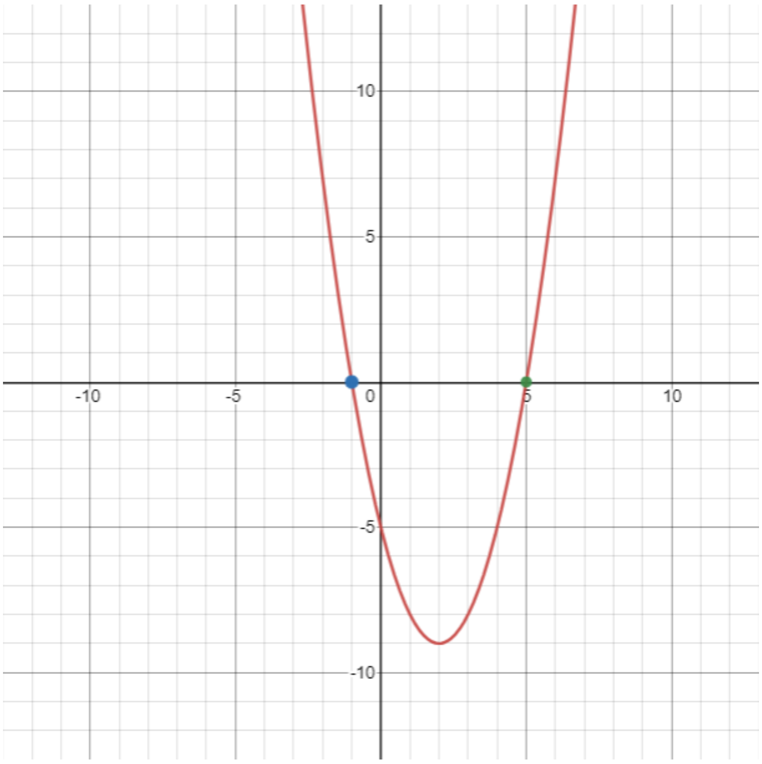

Figure 1: The graph of $f(x) = (x-2)^2 - 9 = 0$

1.1 Questions

1. Solve for x: $5(x-3)^2 - 9 = 11$
2. Solve for x: $-(x+2)^2 + 5 = -76$
3. For which values of x does the function $f(x) = 2(x+1)^2 - 32$ equal 0?
4. For which values of x does the function $f(x) = -2(x+1)^2 - 32$ equal 0?

(Courtesy of KhanAcademy): Quadratics by taking the square root(intro).
(Courtesy of KhanAcademy): Quadratics by taking the square root.
(Courtesy of KhanAcademy): Quadratics by taking the square root(strategy).
(Courtesy of KhanAcademy): Quadratics by taking the square root(with steps).

2 Vertex form

Take a look at these three quadratic equations:

$$y = 3x^2 + 12x - 15,\ y = 3(x-1)(x+5),\ y = 3(x+2)^2 - 27.$$

While they may all seem different, if you expanded them all algebraically you would get the same equation. This is because quadratics have many forms, all of which are helpful when graphing them. For the purposes of this lesson, we're going to focus on the third form, $y = 3(x+2)^2 - 27$, known by mathematicians as vertex form. Firstly, a refresher on the vertex of a parabola:

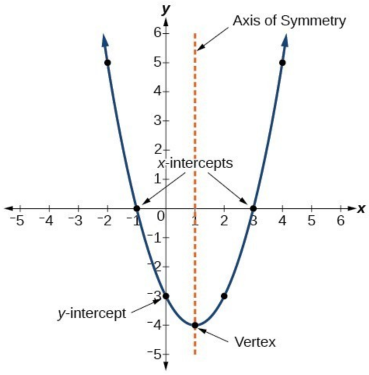

Figure 2: The vertex of a parabola

The vertex of a parabola is either its highest point (in the case of a downwards opening parabola), or its lowest point (in the case of an upwards opening parabola). To determine the direction a parabola opens, there is a very simple trick: If the leading coefficient of the equation is positive, the parabola opens upward, and if the leading coefficient is negative, it opens downwards. For example, the equation

$$y = 7x^2 - 4x - 9$$

graphs to be an upwards facing parabola because the leading coefficient (7) is positive, while the equation

$$y = -(x^2 + 5x + 6)$$

graphs to be a downwards facing parabola because the leading coefficient (-1) is negative. With this in mind, our equation $y = 3(x+2)^2 - 27$ opens upward, because it has a positive leading coefficient. This means that the vertex of the parabola is going to be it's lowest possible point, or the minimum y-value.

Finally, we are now able to appreciate why this form is called vertex form. Observe the $3(x+2)^2$ term of this equation:

$$y = \color{red}{3(x+2)^2}\ -27$$

Note that the $(x+2)^2$ term can never be negative, as the square of a number is always positive. When it's multiplied by 3, it still can't be negative. The smallest possible value of the $3(x+2)^2$ term is 0, when $x = -2$. Thus, the smallest possible y value of the entire function is $0 - 27$ is -27. Through simple observation of the function, we were able to derive it's vertex, which is at $(-2, -27)$.

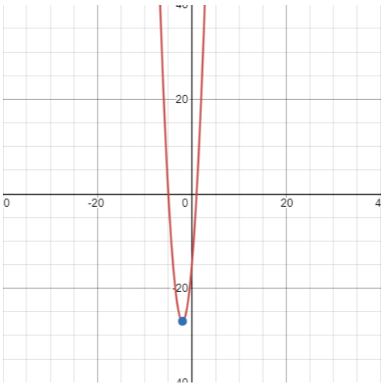

Figure 3: The graph of $y = 3(x+2)^2$

Let's try another example, this time the downward opening parabola of $y = -2(x-5)^2 + 10$. In this case, the vertex occurs at the highest possible y-value. Notice that the $-2(x-5)^2$ can never be negative, so the largest y value occurs when $-2(x-5)^2 = 0$.
At the point where $-2(x-5)^2 = 0$, x would equal 5. Substituting $x = 5$ into the equation gives $y = 10$.
Thus, the vertex of the parabola (or the maximum) is at $(5, 10)$.

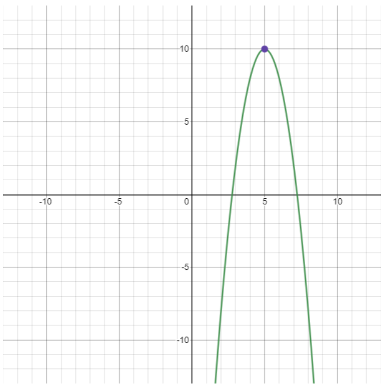

Figure 4: The graph of $y = -2(x-5)^2 + 10$

Now that we are able to derive the vertex by observing the structure of an equation, let's try and generalize it for all vertex-form quadratics. Firstly, let's define vertex form as a quadratic in the following structure:

$$y = a(x-h) + k$$

a is the leading coefficient: If a is positive, the parabola opens upwards, and if a is negative, the parabola opens downward.
h is the x-coordinate of the vertex. In other words, the vertex starts at $(0, 0)$ and moves h units right.
k is the y-coordinate of the vertex. In other words, the vertex starts at $(0, 0)$ and moves k units up.
Through our observations, in vertex form the vertex of the parabola is at (h, k).

2.1 Questions

(Courtesy of KhanAcademy): Graph quadratics in vertex form.
(Courtesy of KhanAcademy): Vertex form word problems. (application)

3 Answer Key

Solving with the square root:

1. 5 or 1
2. 7 or -11
3. 3 or -5
4. No real solutions.