# Logarithm Properties

Logarithms
Algebra II

Julian Zhang

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## 1 Introduction

By now, we know how to evaluate singular logarithms, but we don't know how to solve equations containing multiple of them. We are going to introduce the 4 most important rules of logarithms, upon which all other rules can be derives. Here is a summary of them - we will go through each one, including examples and brief proofs.

**Logarithm Properties:**

$$\log_b(a) + \log_b(c) = \log_b(a \cdot c)$$

$$\log_b(a) - \log_b(c) = \log_b(\frac{a}{c})$$

$$\log_b(a^n) = n \cdot \log_b(a)$$

$$\log_b(a) = \frac{\log_c(a)}{\log_c(b)}$$

## 2 Product Rule

This is the product/sum rule of logarithms, allowing you to convert a sum of two logarithms into one:

**Idea:** Product/Sum rule:

$$\log_b(a) + \log_b(c) = \log_b(a \cdot c)$$

To better conceptualize this, let's use an example:

$$\log_2 8 + \log_2 32$$

We know that $2^3 = 8$, so $\log_2 8 = 3$. We also know that $2^5 = 32$, so $\log_2 32 = 5$. Adding these together, we get

$$\log_2 8 + \log_2 32 = 3 + 5 = 8$$

which is one way to solve this equation. However if we use your sum rule, this gives us

$$\log_2 8 + \log_2 32 = \log_2(8 \cdot 32)$$

$$\log_2 8 + \log_2 32 = \log_2(256)$$

We know that $2^8 = 256$, so we get the same answer of 8, so our property holds true. We're going to prove this in depth later, but our example essentially boils down to

$$2^3 \cdot 2^5 = 2^{(3+5)}$$

This should be quite intuitive, as this is the nature of exponentiation (the inverse of logarithms) - multiplying three 2s, then multiplying that by five 2s is the same as multiplying together $3+5 =$ eight 2s, hence the product rule.

### 2.1 Proof

Let $a$ be $b^x$ and $c$ be $b^y$ for some $x$ and $y$. Converting to logarithm form, this also means

$$\log_b(a) = x$$
and
$$\log_b(c) = y$$

Substituting these values into our formula gives us

$$\log_b(a \cdot c) = \log_b(b^x \cdot b^y)$$

$$\log_b(b^x \cdot b^y) = \log_b(b^{(x+y)})$$

$$\log_b(b^{(x+y)}) = x + y$$

$$x + y = \log_b(a) + \log_b(c)$$

$$\log_b(a \cdot c) = \log_b(a) + \log_b(c)$$

Q.E.D.

### 2.2 Questions

1. Rewrite $\log(5) + \log(2)$ in the form of $\log(c)$
2. Evaluate $\log_6(9) + \log_6(4)$

## 3 Quotient Rule

Likewise, the quotient rule converts from a difference of two logarithms into one:

**Idea:** Quotient/Difference rule:

$$\log_b(a) - \log_b(c) = \log_b(\frac{a}{c})$$

Let's use another example, this time

$$\log_3 729 - \log_3 9$$

We know that $3^6 = 729$, and $3^2 = 9$

$$\log_3 729 - \log_3 9 = 6 - 2 = 4$$

But by the quotient rule,

$$\log_3 729 - \log_3 9 = \log_3(\frac{729}{9})$$

$$\log_3 729 - \log_3 9 = \log_3(81)$$

$$\log_3 729 - \log_3 9 = 4$$

This should also seem intuitive, as this is just a different version of the sum rule: addition is the inverse of subtraction, just as multiplication is the inverse of division. Ultimately, this example boils down to

$$\frac{3^6}{3^2} = 3^{(6-2)}$$

## 3.1 Proof

If we keep our same values of $a = b^x$ and $x = b^y$, we can prove this in a similar way:

$$\log_b(\frac{a}{c}) = \log_b(\frac{b^x}{b^y})$$

$$\log_b(\frac{b^x}{b^y}) = \log_b(b^{(x-y)})$$

$$\log_b(b^{(x-y)}) = x - y$$

$$x - y = \log_b(a) - \log_b(c)$$

$$\log_b(a) - \log_b(c) = \log_b(\frac{a}{c})$$

Q.E.D.

## 3.2 Questions

1. Rewrite $\log(12) - \log(3)$ in the form of $\log(c)$
2. Evaluate $\ln(9e^3) - \ln(9e)$

# 4 Power Rule

**Idea:** Power rule:

$$\log_b(a^n) = n \cdot \log_b(a)$$

Let's use the example of

$$3 \cdot \log_2(8)$$

Evaluating the logarithm separately, we know that $\log_2(8) = 3$, and $3 \cdot 3 = 9$. When we use our formula, it gives us

$$\log_2(8^3)$$

$$= \log_2(512)$$

We know that $\log_2(512) = 9$, confirming our formula.

## 4.1 Proof

This time, we are only provided with one variable. Let $a = b^x$:

$$\log_b(a^n) = \log_b((b^x)^n)$$

$$\log_b((b^x)^n) = \log_b(b^{x \cdot n})$$

$$\log_b(b^{xn}) = x \cdot n$$

$$n \cdot x = n \cdot \log_b(a)$$

$$\log_b(a^n) = n \cdot \log_b(a)$$

Q.E.D.

## 4.2 Questions

1. Rewrite $4 \cdot \log(2)$ in the form of $\log(c)$
2. Evaluate $3 \cdot \log_2(4)$

# 5 Change of Base Rule

This final rule is arguably the most important of the 4, and it is especially useful when trying to evaluate certain logarithms with a calculator:

**Idea:** Change of Base rule:

$$\log_b(a) = \frac{\log_c(a)}{\log_c(b)}$$

Among other applications, the change of base rule is most useful to evaluate logarithms which are not rational powers. Most calculators don't have a $\log_b(x)$ button, rather they only have a log and $\ln(x)$ button, with the bases of 10 and $e$ respectively. In order to evaluate equations in other bases, we must change the base.

Let's say we wanted to evaluate $\log_2(50)$, but our calculator only had a $\log(x)$ button. 50 is not a rational power of 2, so we have to change the base to 10:

$$\log_2(50) = \frac{\log(50)}{\log(2)}$$

Using our calculator, we get that

$$\log_2(50) \approx 5.644$$

## 5.1 Proof

This seems great and all, but why does it work? To prove this, let's keep using the example of

$$\log_2(50) = \frac{\log(50)}{\log(2)}$$

but let $\log_2(50)$ be n. Converting to exponential form, we get that

$$2^n = 50$$

Take the logarithm of both sides:

$$\log(2^n) = \log(50)$$
$$n\log(2) = \log 50$$
$$n = \frac{\log(50)}{\log(2)}$$
$$\log_2(50) = \frac{\log(50)}{\log(2)}$$

Q.E.D.

Note that in the step where we took the logarithm of both sides $\log(2^n) = \log(50)$, we could have used any base, hence why the change of base rule lets you pick any base.

## 5.2 Applications

Aside from evaluating logarithms, you can also use the change of base rule to simplify equations with different bases. For example, what if I asked you to simplify

$$\frac{\log(t)}{\log_8(t)}$$

To start, we can change the base of the denominator to 10:

$$\frac{\log(t)}{\frac{\log(t)}{\log(8)}}$$

This is the same thing as

$$\log(t) \cdot \frac{\log(8)}{\log(t)}$$

The two $\log(t)$s cancel out, leaving us with

$$\log(8)$$

## 5.3 Questions

1. Evaluate $\log_4(\frac{1}{19})$, round to the nearest thousandth
2. Evaluate $\log_5(\frac{1}{1000})$, round to the nearest thousandth
3. Evaluate $\log_3(b) \cdot \log_b(27)$
4. Evaluate $\frac{\log_b(8)}{\log_b(2)}$

# 6 Homework

1. Rewrite $\log(3) + \log(4)$ in the form of $\log(c)$
2. Evaluate $\log(20) + \log(50)$
3. Rewrite $\log(30) - \log(5)$ in the form of $\log(c)$
4. Evaluate $\log_3(324) - \log_3(4)$
5. Rewrite $3 \cdot \log(2)$ in the form of $\log(c)$
6. Evaluate $6 \cdot \log_{16}(4)$
7. Evaluate $3 \cdot \log_9(\frac{1}{12})$, round to the nearest thousandth
8. Evaluate $2 \cdot \log_3(\frac{1}{52})$, round to the nearest thousandth
9. Simplify $\log(a) \cdot \log_a(5)$

# 7 Answer Key

**Product Rule:**

1. log(10)
2. 2

**Quotient Rule:**

1. log(4)
2. 2

**Power Rule:**

1. log(16)
2. 6

**Change of Base Rule:**

1. -2.124
2. -4.292
3. 3
4. 3

**Homework**

1. log(12)
2. 3
3. log(6)
4. 4
5. log(8)
6. 3
7. -3.393
8. -7.193
9. log(5)