The constant e

Logarithms

Algebra II

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Abstract

As we mentioned in the last lesson, mathematicians often use the constant e, or Euler's number as the base of the natural logarithm. But where did this number come from, and why is it so special?

1 History

In the 17^{th} century, mathematician Jacob Bernoulli was interested in compound interest

Figure 1: Jacob Bernoulli (1655-1705)

Imagine you went to a bank, where you wanted to invest \$1. This bank is very generous, and it offers 100% interest every year. This means that after one year, you'll have

$$\$1 + \$1 = \$2$$

Now this sounds great, but what if the bank proposed another offer: instead of offering 100% interest every year, they would offer you 50% interest every half-year, or 6 months. After this first 6 months, you would have

$$\$1 \times 1.5 = \$1.5$$

and after the next 6 months, you would have

$$\$1.5 \times 1.5 = \$2.25$$

This sounds better than the original \$2. Now, this has you thinking, what if we spread out the interest over smaller intervals of time? Imagine the same interest, but instead of every 6 months, it would compound after every month.
How much interest would you have after a year of monthly interest? We could express this as

$$\$1 \times (1 + \frac{1}{12})^{12}$$

meaning that the principle of \$1 adds one-twelfth of itself, and does that twelve times (one per month). Plugging this into our calculator gives us

$$\$1 \times (1 + \frac{1}{12})^{12} = \$2.61$$

Now this sounds even better than the previous \$2.25, further reinforcing our theory that compounding interest over many smaller intervals is better than over just one large interval. Let's test it again, this time with interest over every week. There are 52 weeks in a year, so we can express our interest as

$$\$1 \times (1 + \frac{1}{52})^{52} = \$2.69$$

which is even better than once a month.
At this point, you might notice that the equations for calculating compounding interest in this way follow a certain formula: Interest that is compounded over n intervals can be calculated by the formula

$$(1 + \frac{1}{n})^{n}$$

One last example, but this time, what if we calculated the interest ever day? There are 365 days in a year, so plugging that into our formula gives us

$$(1 + \frac{1}{365})^{365} = \$2.71$$

which is now significantly better than the original \$2.
Since we now know that decreasing the length of the interval increases, what if we had interest given at every minute, second, or even smaller units of time? What if we had interest given at every instant? To express this mathematically (and this introduces some of the core ideas of rate of change and infinitely small intervals seen in calculus), we would write

$$\lim_{n \to \infty} (1 + \frac{1}{n})^{n}$$

which expresses the value of the interest if interest was given at every instant, in infinitely small intervals. Bernoulli tried his entire life, but was never able to find what this limit equals. He knew that it was between 2 and 3, but it was eventually Leonhard Euler who discovered it.

Figure 2: Leonhard Euler (1707-1783)

Euler found that this number, which he called e is equal to approximately 2.71828. He also was able to prove that e was irrational (never ending), and among a plethora of other definitions, is equal to the sum of the infinite series, or

$$e = 1 + \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \dots$$

In summation notation, we would write this as

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

Among other properties, the function $y = e^x$ is very important in calculus, because it is the only function who's derivative is itself: this means that if at any point in the graph of $y = e^x$, the instantaneous rate of change is also e^x.
Also, since the inverse of e^x is $\ln(x)$, their graphs are inverses of each other:

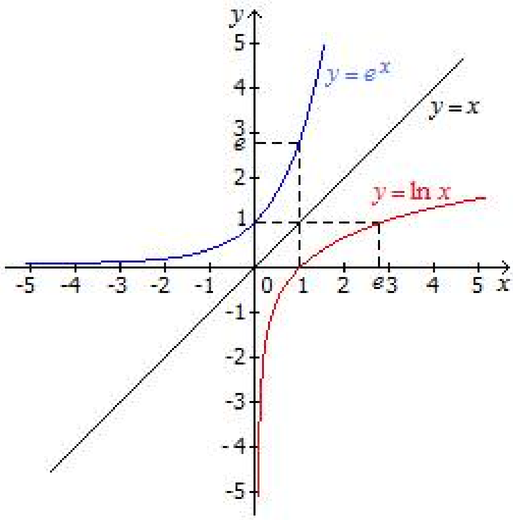

Figure 3: Graphs of $y = e^x$ and $y = \ln(x)$

In conclusion, Euler's constant e remains one of the most important numbers when studying logarithms, and in much more math down the road.

2 Extra Reading

1. Sal Khan - e and compound interest
2. Sal Khan - e as a limit
3. Sal Khan - Evaluating natural logarithm with a calculator
4. Numberphile - e (Euler's Number)
5. 3blue1brown - What's so special about Euler's number e? (advanced)