Intro to Logarithms

Logarithms
Algebra II

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1 Introduction

In mathematics, exponentiation is a shorthand for repeated multiplication. For example, when we write 2^4, this means

$$2^4 = 2 \times 2 \times 2 \times 2$$
$$= 16$$

However, what if we wanted to perform this operation in reverse? Let's say that we needed to find a number such that 2 raised to that power equals 16. In other words, to find a number x such that

$$2^x = 16$$

Just from memory, we would know that x in this case equals 4. However, in the case that we didn't it them by memory, we invented the logarithm (or *log* for short) function to help us find these unknown values. To rewrite this equation, we would say

$$x = \log_2 16$$

This is the basis of logarithm notation that mathematicians use.

Idea: We can rewrite the statement $x = b^a$
as
$\log_b(x) = a$

Terminology:
The exponential equation $x = b^a$ is pronounced "x equals b to the power of a".
Conversely, the logarithmic equation $\log_b(x) = a$, is pronounced "log base b of x equals a". In both equations, we say that:
b is the **base**,
a is the **exponent**, and
x is the **argument**.

To give another example, what if I asked you to evaluate

$$\log_3 81$$

This function essentially asks you to find a number such that 3 raised to that power equals 81. As such, we know that 3^4, or $3 \times 3 \times 3 \times 3 = 81$. Thus,

$$\log_3 81 = 4$$

One last example, this time with

$$\log_{100} 1$$

To convert this into exponential form, this is the same as asking you to find a number x such that

$$100^x = 1$$

We know that in order for a power of a number (aside from 1) to equal 1, it must be raised to the power of 0. Thus, the solution is

$$\log_{100} 1 = 0$$

Now that we understand how to convert between exponential and simple logarithmic form, note that logarithms also have a few restrictions. The equation

$$\log_b a = x$$

is only defined when three things are true:

1. b is positive
2. a is positive
3. b does not equal 1

Here is the reasoning:

1. $b > 0$: In an exponential function, the base b is always defined to be positive.
2. $a > 0$: $\log_b a = x$ means that $b^x = a$. Since a positive number raised to any power is always positive, a is always defined to be positive.
3. $b \neq 1$: Since 1 to the power of anything is 1, the logarithm can never be true, thus b can never equal 1.

1.1 Questions

1. Write $2^5 = 32$ in logarithmic form
2. Write $\log_2 64 = 6$ in exponential form
3. Evaluate $\log_6 36$
4. Evaluate $\log_7 343$
5. Evaluate $\log_4 4$
6. Evaluate $\log_5 1$

2 Special Logarithms

While we have introduced logarithms with a changeable base, there are two main bases that are found on most scientific calculators, and are used more than others.
Firstly, the common logarithm, most commonly written as just $\log(x)$. In mathematics, we usually omit the base, and it is commonly understood to be base 10. The only exception to this rule is in computer science, where $\log(x)$ usually refers to $\log_2(x)$. In short,

$$\log(x) = \log_{10} x$$

Secondly, the natural logarithm, which is a logarithm whose base is the number e. e, or Euler's number is a mathematical constant approximately equal to 2.718. We will learn much more about this constant in the future, but for now, just treat e as you would any other number.
Instead of writing a logarithm with base e, we shorten it to ln.

$$\ln(x) = \log_e x$$

2.1 Questions

1. Evaluate $\log(100)$
2. Evaluate $\ln(e^3)$

3 Evaluating Logarithms (advanced)

By now, we are familiar with the fact that the function $\log_2 8$ asks us for a number such that 2 raised to that power equals 8. We know that 2^3=8, so the answer would be 3.
But what if I asked you to evaluate

$$\log_8 2$$

Converting to exponential form, this is the same as asking you to find a number x such that $8^x = 2$. Since we know that $8 = 2^3$, our x value would not equal 3, but $\frac{1}{3}$. This is the same as writing $\sqrt[3]{8} = 2$ Thus,

$$\log_8 2 = \frac{1}{3}$$

Now, what if I asked you to evaluate

$$\log_2 \frac{1}{8}$$

In this case, we know that 2^3 is 8, but since $\frac{1}{8}$ is the reciprocal of 8, we would have to take the negative:

$$\log_2 \frac{1}{8} = -3$$

Finally, what if I asked you to evaluate

$$\log_8 \frac{1}{2}$$

Since we already know that $8^{\frac{1}{3}} = 2$, we need to take the reciprocal. Thus,

$$\log_8 \frac{1}{2} = -\frac{1}{3}$$

3.1 Questions

1. Evaluate $\log_{49} 7$
2. Evaluate $\log_{16} \frac{1}{2}$
3. Evaluate $\log_{\frac{1}{5}} 5$
4. Evaluate $\log_{81} \frac{1}{27}$

4 Homework

1. Write $5^3 = 125$ in logarithmic form
2. Write $\log_4 16 = 2$ in exponential form
3. Evaluate $\log_4 256$
4. Evaluate $\log_3 \frac{1}{9}$
5. Evaluate $\log(10000)$
6. Evaluate $\ln(e^7)$
7. Evaluate $\log_{216} \frac{1}{36}$
8. Evaluate $\log_{16} 8$

5 Answer Key

Logarithms:

1. $\log_2 32 = 5$
2. $2^6 = 64$
3. 2
4. 3
5. 1
6. 0

Special Logarithms:

1. 2
2. 3

Advanced Logarithms

1. $-\frac{1}{2}$
2. $-\frac{1}{4}$
3. -1
4. $-\frac{3}{4}$

Homework:

1. $\log_5 125 = 3$
2. $2^4 = 16$
3. 4
4. -2
5. 5
6. 7
7. $-\frac{2}{3}$
8. $\frac{3}{4}$