Sequences and Series

Julian Zhang

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1 Introduction

This handout is an introduction to Sequences, Series, and Inequalities covered in AMC 10/12 level math. Credits go to Eric Shen (as always) for the problems.

2 Sequences and Series

A series is defined as a series of numbers in a particular order, and there are two main ones in contest math: arithmetic and geometric. A series is defined as the sum of the first n terms of a sequence, where n can be finite or infinite.

An arithmetic sequence is of the form $a, a+d, a+2d, \ldots$ where each term is equal to the last, plus some common difference d.

A geometric sequence is of the form $a, ar, ar^2, \ldots$ where each term is equal to the last, multiplied by some common ratio r.

The best trick for solving problems which tell you n numbers form arithmetic and geometric sequences is to instead represent it as an algebra problem, using a as the first term and either their common difference or common ratio.

2.1 Sum Quantities

Definition: **Sum of Arithmetic Sequence:** $a, a+d, a+2d, \ldots$

$$\sum_{i=1}^{n} a_i = an + \frac{n(n-1)}{2}d$$

Definition: **Sum of Geometric Sequences:** $a, ar, ar^2, \ldots$

$$\sum_{i=1}^{n} a_i = \frac{a(1-r^n)}{1-r}$$

$$\sum_{i=1}^{\infty} a_i = \frac{a}{1-r}$$

2.2 Derivation of Sum Quantities

Sum of Arithmetic Sequence: We can imagine the sum of the first n terms as $a + (a + d) + (a + 2d) + \dots (a + (n-1)d)$. Each term has one a, and we can use the Gaussian sum from 1 to $n - 1$ for d, giving us

$$S = an + \frac{n(n-1)}{2}d$$

Obviously, the infinite sum of an arithmetic sequence doesn't converge towards anything, so it equals positive or negative infinity (depending on the value of d)

Sum of Infinite Geometric Sequence: Let $a + ar + ar^2 + \dots = S$. We have:

$$\begin{aligned} rS &= ar + ar^2 + ar^2 + \dots \\ s &= a + ar + ar^2 + \dots - a \\ &= S - a \\ (r-1)S &= -a \\ S &= \frac{a}{1-r} \end{aligned}$$

Sum of Finite Geometric Sequence: Let $a + ar + ar^2 + \dots + a^{n-1} = S$. We have:

$$\begin{aligned} rS &= ar + \dots ar^n \\ &= a + ar + \dots ar^{n-1} + ar^n - a \\ &= S + ar^n - a \\ &= S + a(r^n - 1) \\ (r-1)S &= a(r^n - 1) \\ S &= \frac{a(r^n - 1)}{r - 1} \end{aligned}$$

2.3 Sequences Heuristics

1. For recursive sequences, try and look for patterns
2. If no recursive formula is given, try writing a_{n+1} in terms of a_n, a_{n-1}, etc. in inductive fashion.
3. Try to telescope and cancel out series

3 Sigma Notation

3.1 Sigma Properties

Definition: We use the uppercase Greek letter Sigma to denote summation in the following way:

$$\sum_{i=1}^{n} x_i = x_1 + x_2 + \dots x_n$$

Theorem: Sigma Properties:

$$\sum_{k=1}^{n} ca_k = c \sum k = 1^n a_k$$

$$\sum_{k=1}^{n} (a_k + b_k) = \sum k = 1^n a_k + \sum k = 1^n b_k$$

3.2 Sigma Methods

1. By grouping/pairing up (derivation of Gaussian Sum)
2. By elimination (derivation of Geometric Series)

$$s = a + ar + ar^2 + \dots$$
$$-rs = ar + ar^2 + ar^3 + \dots$$

3. By telescoping
4. By recursive counting

Theorem: Common Sequences:

- $1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$ (triangular numbers)
- $1 + 2 + 2^2 + \dots 2^n = 2^{n+1} - 1$ (sum of powers of 2)
- $1 + 3 + 5 + \dots + (2n - 1) = n^2$ (sum of odd numbers)
- $1^2 + 2^2 + 3^2 + \dots + \frac{n(n+1)(2n+1)}{6}$ (sum of squares)
- $1^3 + 2^3 + 3^3 + \dots + (\frac{n(n+1)}{2})^2$ (sum of cubes)

4 Examples

Example 1 (CEMC)

Prove that if a, b, c, and d are four consecutive terms in a geometric sequence, then

$$(b - c)^2 - (c - a)^2 + (d - b)^2 = (a - d)^2$$

This is a perfect example of when it helps to convert a sequence problem to an algebra problem.

Denote $b = ar$, $c = ar^2$, and $d = ar^3$. We have that

$$
\begin{aligned}
&= (ar - ar^2)62 + (ar^2 - a)^2 + (ar^3 - ar)^2 \\
&= a^2r^2 - 2a^2r^3 + a^2r^4 + a^2r^4 - 2a^2r^2 + a^2 + a62r^6 - 2a^2r^4 + a^2r^2 \\
&= a^2 - 2a^2r^3 + a^2r^6 \\
&= (a - ar^3)^2 \\
&\boxed{= (a - d)^2}
\end{aligned}
$$

Example 2 (AIME 1989)

If the integer k is added to each of the numbers 36, 300, and 596, one obtains the squares of three consecutive terms of an arithmetic series. Find k.

Here, we can set the three terms as $a - d$, a, and $a + d$. Squaring the three terms gives us $a^2 - 2ad + d^2$, a^2, and $a^2 + 2ad + d^2$. However, since we don't know what k is, it's probably best to take the differences of these quantities. We have that:

$$
\begin{aligned}
a^2 + 2ad + d^2 - a^2 &= 2ad + d^2 \\
&= 596 - 300 \\
&= 296 \\
2ad - d^2 &= 300 - 36 \\
&= 264 \\
d &= 4 \\
2ad &= 264 + d62 \\
8a &= 264 + 16 \\
&= 280 \\
a &= 35
\end{aligned}
$$

Thus, the middle term (equal to $300 + k$) is also equal to $a^2 = 1225$, giving us that $\boxed{k = 925}$

Example 3 (Math League 2011-12)

What is the average of 2012 consecutive positive integers whose sum is 2012^3?

Recall that first of all, a set of consecutive positive integers is basically an arithmetic sequence with common difference 1. In addition, when we add a finite series of an arithmetic sequence, we have that since we pair all the terms together (smallest with largest and so on) when summing, the sum of an arithmetic sequence with n terms is basically n times the average of all the terms. Thus, here we have that since there are 2012 terms that sum to 2012^3, their average must be $\boxed{2012^2}$.

5 Telescoping

Example 4 (Telescoping example)

Compute the value of

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \ldots + \frac{1}{99100}$$

Note that $\frac{1}{x\cdot(x+1)}$ is equal to $\frac{1}{x}-\frac{1}{x+1}$. This gives us that

$$\begin{aligned}\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\dots\frac{1}{99\cdot 100} &= \frac{1}{1}-\frac{1}{2}+\frac{1}{2}\dots-\frac{1}{100}\\ &= \frac{1}{1}-\frac{1}{100}\\ &= \frac{99}{100}\end{aligned}$$

The main idea of telescoping is that given a sum of a bunch of terms in a sequence, you express each term in the sequence as a difference of two consecutive terms in another sequence (in this case, the sequence where the xth term is $\frac{1}{x}$. By doing this, every single term except for the first and last cancel, giving the answer.

5.1 Partial Fraction Decomposition

One trick that helps with this (and one that we just used) is something called partial fraction decomposition - which for our purposes basically means that if we have a bunch of linear factors in a denominator and one less number of factors in the numerator, we can essentially separate the fraction into a sum/difference of a bunch of fractions.

As an example, we'll take the partial fraction decomposition of $\frac{3x+5}{(x+1)(x+2)}$.

We want this to represent

$$\frac{3x+5}{(x+1)(x+2)}=\frac{A}{x+1}+\frac{B}{x+2}$$

for some A and B. Multiplying by $(x+1)(x+2)$ gives

$$\begin{aligned}3x+5 &= A(x+1)+B(x+2)\\ 3 &= A+B\\ 5 &= A+2B\end{aligned}$$

This gives us the solution of $(A,B)=(1,2)$. Thus, we have that

$$\frac{3x+5}{((x+1)(x+2)}=\frac{1}{x+1}+\frac{2}{x+2}$$

6 Inequalities

Definition: Mean Quantities

- The arithmetic mean of n numbers $a_1,a_2,\dots,a_n$,

$$A(a)=\frac{a_1+a_2+\dots a_n}{n}$$

- The geometric mean of n nonnegative real numbers,

$$G(a)=\sqrt[n]{\frac{1}{a_1}+\frac{1}{a_2}+\dots\frac{1}{a_n}}$$

- The square mean of n real numbers,

$$S(a) = \frac{\sqrt{a_1^2 + a_2^2 + \ldots + a_n^2}}{n}$$

- The harmonic mean of n real numbers,

$$H(a) = \frac{1}{\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}}$$

We then have the following relationships:

- $A(a) \geq G(a)$ for non-negative real numbers (AM-GM inequality)
- $S(a) \geq A(a)$ for real numbers
- $G(a) \geq H(a)$ for positive real numbers

Theorem: AM-GM:

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \cdots x_n}$$

NOTE: This is a special case of Jensen's Inequality

Theorem: Cauchy:

$$(a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2) \geq (a_1 b_1 + a_2 b_2 + \cdots + a_n b_n)^2,$$

NOTE: This is a special case of Holder's Inequality

7 Exercises

1. What is the value of the product

$$\left(1 + \frac{1}{1}\right) \cdot \left(1 + \frac{1}{2}\right) \cdot \left(1 + \frac{1}{3}\right) \cdot \left(1 + \frac{1}{4}\right) \cdot \left(1 + \frac{1}{5}\right) \cdot \left(1 + \frac{1}{6}\right)?$$

(A) $\frac{7}{6}$ **(B)** $\frac{4}{3}$ **(C)** $\frac{7}{2}$ **(D)** 7 **(E)** 8

2. The first three terms of a geometric progression are $\sqrt{3}$, $\sqrt[3]{3}$, and $\sqrt[6]{3}$. What is the fourth term?

(A) 1 **(B)** $\sqrt[7]{3}$ **(C)** $\sqrt[8]{3}$ **(D)** $\sqrt[9]{3}$ **(E)** $\sqrt[10]{3}$

3. Suppose that $\{a_n\}$ is an arithmetic sequence with

$$a_1 + a_2 + \cdots + a_{100} = 100 \text{ and } a_{101} + a_{102} + \cdots + a_{200} = 200.$$

What is the value of $a_2 - a_1$?

(A) 0.0001 (B) 0.001 (C) 0.01 (D) 0.1 (E) 1

4. Let $a_1, a_2, \cdots, a_k$ be a finite arithmetic sequence with $a_4 + a_7 + a_{10} = 17$ and $a_4 + a_5 + \cdots + a_{13} + a_{14} = 77$.

   If $a_k = 13$, then $k =$

   (A) 16 (B) 18 (C) 20 (D) 22 (E) 24

5. Find the value of $a_2 + a_4 + a_6 + a_8 + \ldots + a_{98}$ if $a_1, a_2, a_3 \ldots$ is an arithmetic progression with common difference 1, and $a_1 + a_2 + a_3 + \ldots + a_{98} = 137$.

6. For $-1 < r < 1$, let $S(r)$ denote the sum of the geometric series

$$12 + 12r + 12r^2 + 12r^3 + \cdots.$$

   Let a between -1 and 1 satisfy $S(a)S(-a) = 2016$. Find $S(a) + S(-a)$.

7. Let $a_1, a_2, \ldots$ be an arithmetic sequence and $b_1, b_2, \ldots$ be a geometric sequence. Suppose that $a_1b_1 = 20$, $a_2b_2 = 19$, and $a_3b_3 = 14$. Find the greatest possible value of a_4b_4.

8. In an increasing sequence of four positive integers, the first three terms form an arithmetic progression, the last three terms form a geometric progression, and the first and fourth terms differ by 30. Find the sum of the four terms.

9. Arithmetic sequences (a_n) and (b_n) have integer terms with $a_1 = b_1 = 1 < a_2 \leq b_2$ and $a_nb_n = 2010$ for some n. What is the largest possible value of n?

   (A) 2 **(B)** 3 **(C)** 8 **(D)** 288 **(E)** 2009

10. Let $ABCD$ be a unit square. Let Q_1 be the midpoint of $\overline{CD}$. For $i = 1, 2, \ldots,$ let P_i be the intersection of $\overline{AQ_i}$ and $\overline{BD}$, and let Q_{i+1} be the foot of the perpendicular from P_i to $\overline{CD}$. What is

$$\sum_{i=1}^{\infty} \text{Area of } \triangle DQ_iP_i\,?$$

   (A) $\frac{1}{6}$ **(B)** $\frac{1}{4}$ **(C)** $\frac{1}{3}$ **(D)** $\frac{1}{2}$ **(E)** 1

11. Evaluate $\sum_{k=2}^{n} k!(k^2 + k + 1)$.

12. Compute

$$\sum_{n \geq 1} \frac{(7n + 32) \cdot 3^n}{n \cdot (n + 2) \cdot 4^n}$$