Trigonometry

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1 Introduction

This handout is designed to be a comprehensive list of all trigonometric concepts and formulas needed for pre-Olympiad high school contest math. This handout is written with the assumption that the reader understands the basic trigonometric functions and their usages, as taught in the Ontario Grade 9-11 curriculum. Credits go to George Wang and AOPS Volume 2 for the questions.

2 Trig Identities

Definition: The three main trigonometric identities and their inverses:

$$\sin(x) = \frac{opp}{hyp} \qquad \csc(x) = \frac{1}{\sin(x)}$$
$$\cos(x) = \frac{adj}{hyp} \qquad \sec(x) = \frac{1}{\cos(x)}$$
$$\tan(x) = \frac{opp}{adj} \qquad \cot(x) = \frac{1}{\tan(x)}$$

2.1 Even-odd Identities

$$\sin(-x) = -\sin(x)$$
$$\cos(-x) = \cos(x)$$
$$\tan(-x) = -\tan(x)$$

2.2 Period Identities

$$\sin(x \pm 2\pi) = \sin(x)$$
$$\cos(x \pm 2\pi) = \cos(x)$$
$$\tan(x \pm \pi) = \tan(x)$$
$$\csc(x \pm 2\pi) = \csc(x)$$
$$\sec(x \pm 2\pi) = \sec(x)$$
$$\cot(x \pm \pi) = \cot(x)$$

2.3 Conversion Identities

$$\cos(\frac{\pi}{2} - x) = \sin(x)$$
$$\sin(\frac{\pi}{2} - x) = \cos(x)$$
$$\tan(\frac{\pi}{2} - x) = \tan(x)$$
$$\cot(\frac{\pi}{2} - x) = \tan(x)$$
$$\csc(\frac{\pi}{2} - x) = \sec(x)$$
$$\sec(\frac{\pi}{2} - x) = \csc(x)$$

2.4 Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1$$
$$\tan^2\theta + 1 = sec^2\theta$$
$$\cot^2\theta + 1 = \csc^2\theta$$

2.5 Sum and Difference Formulas

$$\sin(x \pm y) = \sin(x)\cos(y) \pm \cos(x)\sin(y)$$
$$\cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x) sin(y)$$
$$tan(x \pm y) = \frac{\tan(x) \pm \tan(y)}{1 \mp \tan(x)\tan(y)}$$

2.6 Product to Sum formulas

$$\sin(x)\sin(y) = \frac{1}{2}[\cos(x-y) - \cos(x+y)]$$
$$\cos(x)\cos(y) = \frac{1}{2}[\cos(x-y) + cos(x+y)]$$
$$sin(x)\cos(y) = \frac{1}{2}[\sin(x+y) + sin(x-y)]$$

2.7 Sum to Product formulas

$$\sin x \pm \sin y = 2\sin\frac{x \pm y}{2}\cos\frac{x \mp y}{2}$$
$$\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$$
$$\cos x - \cos y = -2\sin\frac{x+y}{2}\sin\frac{x-y}{2}$$

2.8 Double-angle formulas

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$$
$$cos(2\theta) = cos^2(\theta) - \sin^2(\theta) = 1 - 2\sin^2(\theta) = 2\cos^2(\theta) - 1$$
$$\tan(2\theta) = \frac{2\tan(\theta)}{1 - \tan^2(\theta)}$$

2.9 Half-angle formulas

$$\sin(\frac{x}{2}) = \pm\sqrt{\frac{1-\cos(x)}{2}}$$
$$\cos(\frac{x}{2}) = \pm\sqrt{\frac{1+\cos(x)}{2}}$$
$$\tan(\frac{x}{2}) = \frac{1-\cos(x)}{\sin(x)}$$

2.10 Function Laws

For all following formulas, assume we have a triangle ΔABC with side a opposite angle A, side b opposite angle B, and side c opposite angle C:

Law of Sines:

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$$

Law of Cosines:

$$a^2 = b^2 + c^2 - 2bc\cos(A)$$

Law of Tangents (obscure):

$$\frac{a-b}{a+b} = \frac{\tan[\frac{1}{2}(A-B)]}{\tan[\frac{1}{2}(A+B)]}$$

2.11 Area of Triangles

$$[ABC] = \frac{1}{2}ab\sin(C)$$
$$[ABC] = \sqrt{s(s-a)(s-b)(s-c)}$$

2.12 Misc Formulas

Amplitude Moderation: $a\sin x + b\cos x = \sqrt{a^2+b^2}\sin(x+\alpha) = \sqrt{a^2+b^2}\cos(x-\beta)$

3 Graphing

4 Exercises

1. Find side AC of ΔABC if $\angle A = 90°$, $\sec(B) = 4$, and $AB = 6$.
2. Find, in degrees, the smallest positive angle x such that $\sin(3x) = \cos(7x)$
3. Evaluate $\sin(75°)$ without a calculator.
4. Find the value of $\sin^2 10° + \sin^2 20° + \ldots + \sin^2 90°$
5. Show that $\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$ using the formula for $\sin(\alpha - \beta)$
6. From the top of a fire tower, a forest ranger sees his partner on the ground at an angle of depression of 40°. If the tower is 45 feet in height, how far is the partner from the base of the tower, to the nearest tenth of a foot?
7. If $\sin(x)\cos(x) = \sqrt{22}$, find x.
8. Evaluate $\cos(36°) - \cos(72°)$ without a calculator.