Number Theory

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1 Introduction

This handout is designed to be an guide for most AMC 10/12 level number theory, with the exception of mods (which will be covered in a later handout). Some of these concepts could manifest on their own, but number theory is likely better applied as a way of modelling algebra or other problems. As always, thanks to Eric Shen.

2 Bases and Basics

Definition: In base n, the largest possible value of a digit is $n-1$. A number x in base n is written as x_n, and the numerical value of $\overline{a_k a_{k-1} \cdots a_0}$ is

$$a_k n^k + a_{k-1} n^{k-1} + \cdots a_0$$

The number system we use (digits 0-9) is called decimal, or base 10. When you hear the word *binary* for instance, that refers to base 2. The number 1001100_2 is equal to

$$0 \times 2^0 + 0 \times 2^1 + 1 \times 2^2 + 1 \times 2^3 + 0 \times 2^4 + 0 \times 2^5 + 0 \times 2^6$$

$$4 + 8 + 64 = 76$$

In bases higher than 10, letters are used in place for numbers. In hexadecimal (base-16), $A = 10, B = 11, \cdots F = 15$. For example, the number ABC_{16} is equal to

$$12 \times 16^0 + 11 \times 16^1 + 10 \times 16^2$$

$$12 + 167 + 2560 = 2739$$

Aside from bases, the only other things you really need to know are divisibility rules. Try to have the basics memorized (the rules for 2-10, excluding 7 since there isn't a pretty one). In case you didn't know, the divisibility rule for 11 is that the difference between the sums of alternating digits should be divisible by 11.

3 Divisibility, GCD, and LCM

For everything that follows to hold true, we must first introduce a pretty fittingly-named theorem:

Theorem: Fundamental Theorem of Arithmetic:

Every integer greater than 1 either is a prime itself or is the product of prime numbers. This product is unique up to the reordering of the factors. The general form is written as

$$\prod_{i=1}^{n} p_i^{e_i}$$

where p_i are distinct primes and e_i are nonnegative integers.

In case you're interested, this is proven by strong induction on n. Otherwise, the implications of this are pretty useful. The proofs should seem fairly intuitive.

1. n has $(a_1+1)(a_2+1)\cdots(a_n+1)$ factors.
2. n has $(a_1+1)(a_2+1)\cdots(a_n+1)$ perfect square factors (this pattern continues)
3. If the product in part 1 is equal to k, the then product of all the factors of n is $n^{\frac{k}{2}}$. This is achieved by pairing the corresponding factors together.

More importantly though, this allows us to calculate the GCD and LCM of numbers. You probably already know that $gcd(a,b)$ is the largest number that divides a and b, and $lcm(a,b)$ is the smallest number that is divisible by a and b, but they can also be written in the following way:

Definition: Let $a = \prod_{i=1}^{n} p_i^{d_i}$, and $b = \prod_{i=1}^{n} p_i^{e_i}$, then

$$gcd(a,b) = \prod_{i=1}^{n} p_i^{min(d_i,e_i)}$$

$$lcm(a,b) = \prod_{i=1}^{n} p_i^{max(d_i,e_i)}$$

In my opinion, it's better to conceptualize GCD and LCM in terms of prime factors: the GCD is obtained by prime factoring both numbers and taking the lower of each prime factor, while the LCM is obtained by taking the highest. It's also pretty useful to note that

$$\gcd(a,b)(a,b) = ab$$

This means that when you are given two numbers and one of their LCM or GCD, it's easier to figure other the other. It's also basically proven by definition, since choosing multiplying the minimum factor by the maximum factor simply equals the product of both factors.

3.1 Euclidean Algorithm

In addition, an easy way to calculate the gcd of two numbers x and y is the Euclidean Algorithm, where you basically divide the smaller number out of the larger and replace it with the remainder until one number divides the other, like as follows:

$$\gcd(25,65) = \gcd(25, 2\cdot 25 + 10)$$

$$= \gcd(25, 10)$$
$$= \gcd(10 \cdot 2 + 5, 10)$$
$$= \gcd(5, 10)$$
$$= 5$$

This is great and all, but the more useful application of the Euclidean Algorithm is to solve Diophantine Equations.

Definition: A Linear **Diophantine Equation** comes in the form

$$ax + by = c$$

given that a, b, and c are nonzero integers.

These are impossible to solve algebraically, but when you only want to solve for integer solutions, these equations can be modelled as number theory problems. Let's illustrate this with an example: Solve the linear Diophantine equation

$$385x - 1183y = 294$$

This is actually reductable, so dividing everything by gcd(385,1183)=7 yields

$$55x - 169y = 42$$

To start, let's solve $55x - 169y = 1$. The reason for this being that it works better with the algorithm, and that we can always multiply the x and y values by the c value (in this case, 42). We now use the Euclidean Algorithm to find the GCD of our a and b values (55 and 169)

$$169 = 3 \times 55 + 4$$
$$55 = 13 \times 4 + 3$$
$$4 = 1 \times 3 + 1$$

Well, they're actually coprime. But that doesn't matter for our interests, since now we use back-substitution. This is just basically working in reverse, and note how we're not even really changing the equation, instead just rearranging the numbers and keeping the $= 1$.

$$4 = 1 \times 3 + 1 \implies 4 - 1 \times 3 = 1$$
$$4 = 1 \times (55 - 3 \times 4) = 1 \implies 14 \times 4 = 1 \times 55 = 1$$
$$14 \times (169 - 3 \times 55) - 1 \times 55 = 1 \implies 14 \times 169 - 43 \times 55 = 1$$

And now, we have that $(-43, -14)$ is a solution to $55x - 169y = 1$. To solve our original equation, we simply multiply both by 42 to get $(-1806, -588)$. But we're not done yet, because that's only one solution. The set of all solutions is actually

$$(x, y) = (-1806 + 169k, -58 + 55k), k \in \mathbb{Z}$$

Example 1 (Annoying Euclid)

Compute gcd(31463, 9782) and express it as an integer linear combination of 31463 and 9782.

Example 2 (More Annoying Euclid)

Find all pairs of integers (x, y) with $x \geq 1000, y \geq 1000$ such that $726x - 578y = 324$.

4 Factoring

Remember what I said about using number theory to model algebra problems? Yeah, more of that:

Example 3 (2007 AMC10A)

How many ordered pairs (m, n) of positive integers, with $m \geq n$, have the property that their squares differ by 96?
(A) 3 (B) 4 (C) 6 (D) 9 (E) 12

Solution: We have:

$$m^2 = n^2 = (m-n)(m+n)$$
$$= 96 = 2^5 \cdot 3$$

Note that since m n and m + n have the same parity and their product is even, they must both be even as well. Thus, we consider $x = \frac{m+n}{2}$ and $y = \frac{m-n}{2}$. We have that $xy = \frac{2^5 \cdot 3}{4} = 2^3 \cdot 3$. This has $4 \cdot 2 = 8$ factors, or 4 pairs of factors. Since $x \geq y$, we have that there are 4 possibilities for m and n, thus our answer is $\boxed{B}$.

A factoring trick that can sometimes be useful is **Simon's Favourite Factoring Trick**, which states that

$$(x+p)(y+q) = xy + py + xq + pq$$

This is basically just FOIL, but it works well in number theory equations, especially when p and q are integers. Note that most of the time you will have to induce the pq term (add it to both sides), an example being the following question:

Example 4 (Putnam 2018)

Find all ordered pairs (a, b) of positive integers for which

$$\frac{1}{a} + \frac{1}{b} = \frac{3}{2018}$$

Solution: Multiply by 2018ab and factor:

$$2018b + 2018a = 3ab$$

$$9ab - (3 \cdot 2018)a - (3 \cdot 2018)b = 0$$

$$(3 \cdot 3)ab - (3 \cdot 2018)a - (3 \cdot 2018)b + (2018 \cdot 2018) = 20182$$

$$(3a - 2018)(3b - 2018) = 20182$$
$$= 2^2 \cdot 1009^2$$

Both $3a - 2018$ and $3b - 2018$ must be at least -2015 in order for a and b to be positive integers. However, if both factors are negative then one must be less than -2018, a contradiction. Thus both factors must be positive, and we can now go through the (2+1)(2+1) = 9 factors to find that $(673, 1358114)$, $(674, 340033)$, $(1009, 2018)$, and their "flipped" counterparts are the only 6 solutions.

5 Questions

1. For how many positive integers n does $1 + 2 + ... + n$ evenly divide from $6n$?

   (A) 3 (B) 5 (C) 7 (D) 9 (E) 11

2. Positive integers a and b are such that the graphs of $y = ax + 5$ and $y = 3x + b$ intersect the x-axis at the same point. What is the sum of all possible x-coordinates of these points of intersection?

   (A) -20 (B) -18 (C) -15 (D) -12 (E) -8

3. Using the digits 1, 2, 3, 4, 5, 6, 7, 8, 9, form 4-two 4-digit prime numbers, using each digit only once. What is the sum of the 4 prime numbers?

   (A) 150 (B) 160 (C) 170 (D) 180 (E) 190

4. In base 10, the number 2013 ends in the digit 3. In base 9, on the other hand, the same number is written as $(2676)_9$ and ends in the digit 6. For how many positive integers b does the base-b-representation of 2013 end in the digit 3?

   (A) 6 (B) 9 (C) 13 (D) 16 (E) 18

5. The doubling sum function is defined by

$$D(a, n) = \overbrace{a + 2a + 4a + 8a + ...}^{\text{n terms}}.$$

   For example, we have

$$D(5, 3) = 5 + 10 + 20 = 35$$

   and

$$D(11, 5) = 11 + 22 + 44 + 88 + 176 = 341.$$

   Determine the smallest positive integer n such that for every integer i between 1 and 6, inclusive, there exists a positive integer a_i such that $D(a_i, i) = n$.

6. How many nonnegative integers can be written in the form

$$a_7 \cdot 3^7 + a_6 \cdot 3^6 + a_5 \cdot 3^5 + a_4 \cdot 3^4 + a_3 \cdot 3^3 + a_2 \cdot 3^2 + a_1 \cdot 3^1 + a_0 \cdot 3^0$$

   where $a_i \in -1, 0, 1$ for $0 \le i \le 7$?

   (A) 512 (B) 729 (C) 1094 (D) 3281 (E) 59,048

7. How many ordered pairs (m, n) of positive integers are solutions to

$$\frac{4}{m} + \frac{2}{n} = 1?$$

   (A) 1 (B) 2 (C) 3 (D) 4 (E) more than 6

8. How many positive two-digit integers are factors of $2^{24} - 1$

   (A) 4 (B) 8 (C) 10 (D) 12 (E) 14

9. Both roots of the quadratic equation $x^2 - 63x + k = 0$ are prime numbers. The number of possible values of k is

10. The number n can be written in base 14 as $\underline{a}\ \underline{b}\ \underline{c}$ can be written in base 15 as $\underline{a}\ \underline{c}\ \underline{b}$, and can be written in base 6 as $\underline{a}\ \underline{c}\ \underline{a}\ \underline{c}$, where $a > 0$. Find the base-10 representation of n.

11. Let $f(n)$ denote the sum of all the distinct prime divisors of n. For example, $f(120) = 2 + 3 + 5 = 10$. What is the value of $f(f(3^{12} - 1))$?

12. Determine all integer values of n for which $n^2 + 6n + 24$ is a perfect square.

13. Find the least positive integer such that when its leftmost digit is deleted, the resulting integer is $\frac{1}{29}$ of the original integer.

14. How How many ordered pairs (a, b) of positive integers satisfy the equation

$$a \cdot b + 63 = 20 \cdot (a, b) + 12 \cdot \gcd(a, b),$$

where $\gcd(a, b)$ denotes the greatest common divisor of a and b, and (a, b) denotes their least common multiple?

(A) 0 (B) 2 (C) 4 (D) 6 (E) 8

15. How many ordered triples (x, y, z) of positive integers satisfy $(x, y) = 72, (x, z) = 600$, and $(y, z) = 900$?

(A) 15 (B) 16 (C) 24 (D) 27 (E) 64

16. Call a positive integer n k-pretty if n has exactly k positive divisors and n is divisible by k. For example, 18 is 6-pretty. Let S be the sum of positive integers less than 2019 that are 20-pretty. Find $\frac{S}{20}$.

17. There is a prime number p such that $16p + 1$ is the cube of a positive integer. Find p.

18. Assume that a, b, c, and d are positive integers such that $a^5 = b^4, c^3 = d^2$, and $c - a = 19$. Determine $d - b$.

19. Find the number of ordered triplets (a, b, c) of positive integers such that $a < b < c$ and $abc = 2008$.

20. Find the number of positive integers that are divisors of at least one of $10^{10}, 15^7, 18^{11}$

6 Answer Key

1. b
2. e
3. e
4. c
5. 9765
6. d
7. d
8. d
9. b
10. 925
11. 7
12. 4, -2, -4, 10
13. 725
14. b
15. a
16. 472
17. 307
18. 757
19. 3
20. 435