Complex Numbers

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Today's just gonna be a fun lecture on Complex Numbers. Note that a majority of the material for today is taken from the corresponding chapter in AoPS' Volume 2.

1 Basics

First, we'll talk about the imaginary numbers. We've always talked about how at least for reals, $x^2 \geq 0$. However, for what number then say would we have $x^2 = -1$? Turns out, we define this as i (and $-i$ works as well) - where i is our unit **imaginary number**. After this, multiplying i by any real number gives us our range of imaginary numbers.

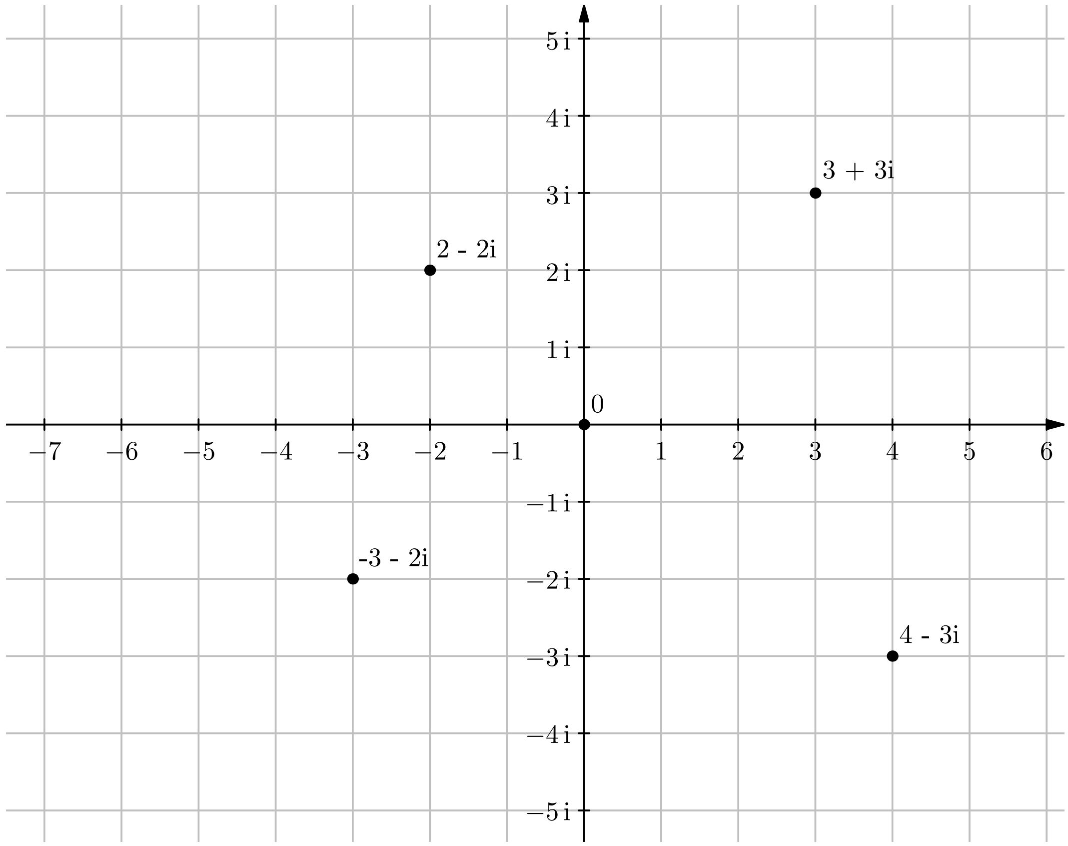
So what then are complex numbers? They're linear combinations of reals and imaginary numbers. For example, $3 + 5i$ is a complex number, and so is $\sqrt{2} + \pi i$. We usually describe imaginary numbers by the combination $a + bi$, where a and b are both real numbers.

Adding complex numbers is done just by adding the real parts in conjunction with the imaginary parts. For example, $(3 + 5i) + (2 + 6i) = 5 + 11i$. Subtraction is also the same manner, and multiplication is also what you would expect - just multiply it out, using FOIL and the fact that $i^2 = -1$. For example, we have:

$$\begin{aligned}(3 + 5i)(2 + 6i) &= 6 + 10i + 18i + 30(i^2) \\ &= 6 + 28i - 30 \\ &= -24 + 28i\end{aligned}$$

Division is a bit different, so I won't teach it (if you want to do it yourself, just know that you have to multiply the denominator by the conjugate to get your answer).

A common way of describing a complex number is through the realization of the complex plane. The complex plane is essentially a cartesian plane that has the "imaginary axis", with unit i, as its y-axis, and the real numbers occupying the x-axis.



Your usual complex plane, with a few example points

2 Conjugates

For any complex number $z = a + bi$, we define its conjugate as the value $\overline{z} = a - bi$. Essentially when interpreted through the complex plane, $\overline{z}$ is the reflection of z across the real axis.

Why the conjugate is important is this:

> The conjugate is preserved through addition, and multiplication. That is - the sum of two conjugates is the conjugate of their sum, and the product of two conjugates is the conjugate of their product.

Note that subtraction is just negative addition anyways so subtraction also works here.

The proof is essentially just to multiply/add it all out - you can do this, but it's not much fun. Why this is important, however, is because addition and multiplication together cover **a lot of stuff**. In particular, one thing we've studied that is just addition and multiplication applied repeatedly to one variable is the polynomial, which we'll investigate now:

Consider a polynomial $P(z)$ that has real coefficients, and consider any complex number z that is a root of this polynomial. Note that we have for any a and nonnegative integer n,

$$\begin{aligned}\overline{z}^n &= \overline{z^n}\\ a\overline{z}^n &= \overline{az^n}\end{aligned}$$

Thus, we have that if $P(z) = a_n z^n + a_{n-1}z^{n-1} + \ldots a_0 z^0 = 0$, then

$$\begin{aligned}P(\overline{z}) &= a_n\overline{z}^n + a_{n-1}^{n-1} + \ldots a_0\overline{z}^0\\ &= \overline{a_n z^n} + \overline{a_{n-1}z^{n-1}} + \ldots \overline{a_0 z^0}\\ &= \overline{P(z)}\\ &= 0\end{aligned}$$

Thus, if z is a root of $P(z)$ then so is $\overline{z}$. This is why I said earlier that complex number roots of a polynomial come in pairs - if we know a complex number is a root of a polynomial, we immediately get a second root for free!

3 Polar Form

A useful technique regarding Complex Numbers is their easy manipulation in polar form. Polar form is basically expressing a complex number $z = a + bi$ not in terms of a and b, but in terms of its absolute value: which we define to be $r = |z| = \sqrt{a^2 + b^2}$, and the angle $\theta = \arg z$ that z makes with the origin of the complex plane. We can readily see that here,

$$\begin{aligned}a &= r\cos\theta\\ b &= r\sin\theta\end{aligned}$$

Why we do this is that multiplication with complex numbers simplifies really nicely with the polar form. Consider two complex numbers $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_1)$. We have that:

$$\begin{aligned}z_1z_2 &= r_1r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)\\ &= r_1r_2(\cos\theta_1\cos\theta_2 + i\sin\theta_1\cos\theta_2 + i\sin\theta_2\cos\theta_1 - \sin\theta_1\sin\theta_2\\ &= r_1r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))\end{aligned}$$

$$\begin{aligned}|z_1z_2| &= r_1r_2\\ &= |z_1||z_2|\\ \arg z_1z_2 &= \theta_1 + \theta_2\\ &= \arg z_1 + \arg z_2\end{aligned}$$

This is a very unique result - what this says is that when we multiply complex numbers in polar form, we multiply the absolute values and add their arguments - in other words, angles! This is a main reason why complex numbers are so unique - their multiplication is very easy to interpret and manipulate.

Fun fact: Euler proved that given any z with absolute value r and argument θ, we can actually express it as $z = re^{i\theta}$, where θ is in radians. In particular, setting $\theta = \pi$ and $r = 1$ gives us that $e^{i\pi} = -1$, or in other words $e^{i\pi} + 1 = 0$ - the formula everyone knows from that guy wearing it on his shirt. On contest or in practice both this form and the polar form we've been using are perfectly valid - in fact, I'll be using this for the rest of the lecture.

In fact, the above result can be repeated multiple times to encompass exponentation as well. We have that for any complex number $z = r(\cos\theta + i\sin\theta)$,

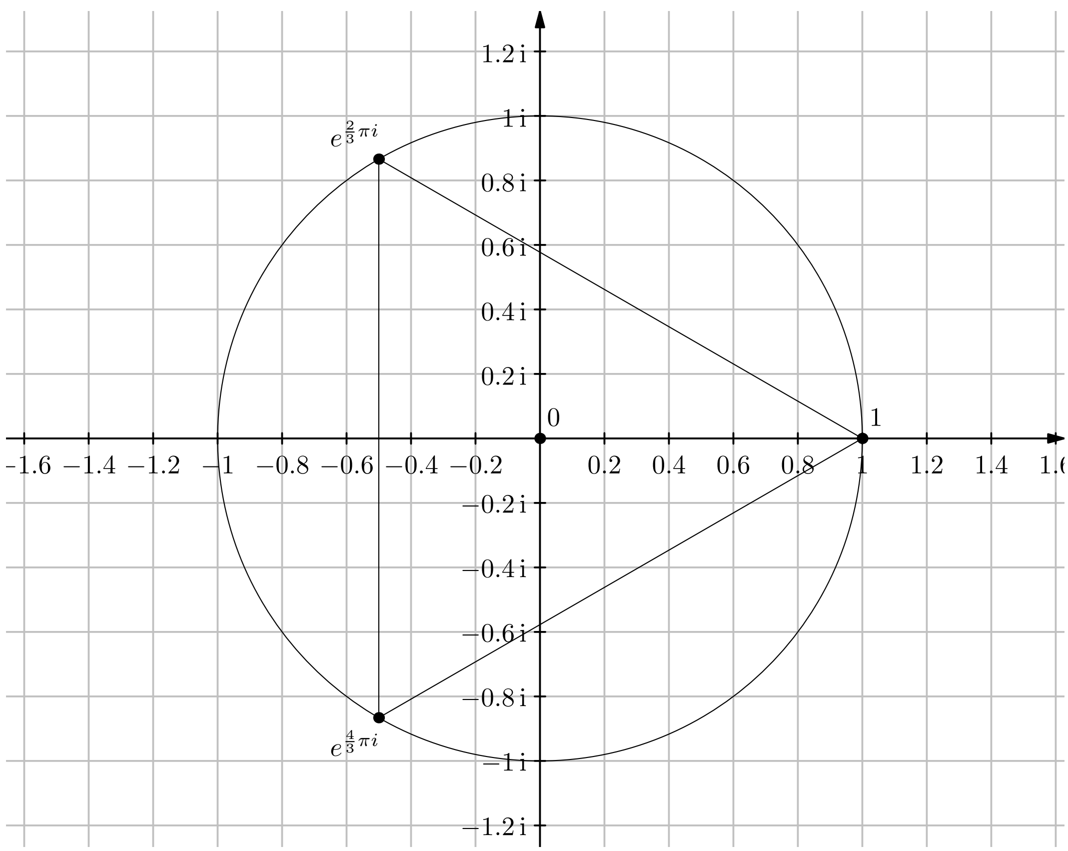
$$\begin{aligned} |z^n| &= |z|^n \\ &= r^n \\ \arg z^n &= n \cdot \arg z \\ &= n\theta \end{aligned}$$

This result is known as **De Moivre's Theorem**, and can prove quite useful on a variety of problems.

4 Roots of Unity

Remember that equation $x^3 = 1$? For so long, we've been taught that $x = 1$ was the only solution. But that's not actually true - in fact, by the Fundamental Theorem of Algebra we know we're missing two complex solutions. So what are they?

Let one of these complex solutions be $z = e^{i\theta}$ We have that $z^3 = e^{i(3\theta)} = 1 = e^{2\pi i}$. This gives us that 3θ is divisible by 2π - giving us that $\theta = \frac{2}{3}\pi$ and $\theta = \frac{4}{3}\pi$ are the two solutions that give us the complex numbers we were looking for. Now, if we plot this in the complex plane, we get the following diagram:



Notice anything?

In fact, we can generalize this to all $x^n = 1$ solutions. The solutions to this equation, called the **nth roots of unity**, are the complex numbers $e^{i\theta}$ for $\theta = \frac{2k}{n}\pi$ where k is an integer from 1 to n inclusive and everything is in radians. In particular, these complex numbers when plotted form a regular n-gon - and perhaps even more surprisingly, this is true for the solutions of this equation for all numbers.

Roots of unity are important just to know, because they appear a lot. Also, their graphic representation is pretty nice, so there's that too :)

5 Examples

Here are a few examples of complex numbers appearing in various contest problems:

1. (AMC 12A 2017) There are 24 different complex numbers z such that $z^{24} = 1$. For how many of these is z^6 a real number?

   (A) 0 **(B)** 4 **(C)** 6 **(D)** 12 **(E)** 24

   Let's consider any such $z = e^{i\theta}$, and we'll look into the value of θ. Note that the condition above gives us that z is a 24th root of unity, and that we want to find all z such that it's also a 6th root of unity. Translated into numerical terms, we have;

   $$\theta = \frac{k}{12}$$

   for some integer k betwee 0 and 24, and we want to find all such k such that we also express θ as:

   $$\theta = \frac{a}{3}$$

   for some integer a. Clearly this is for all k divisible by 4, so our answer is $\frac{24}{4} = 6$, or $\boxed{\text{C}}$.

   This one features a cool technique.

2. (Autumn Mock AMC 10 2018) How many of the solutions to $x^1 + x^2 + \ldots x^{59} = 0$ are not solutions to $x^2 + x^4 + x^6 + x^8 = 0$.

   First of all, multiplying the first equation by $x - 1$ and factoring out the x gives us that:

   $$x(1 + x + \ldots x^{58}) = 0$$
   $$x\frac{x^{59} - 1}{x - 1} = 0$$

   From this we see that the solutions to the first equation are $x = 0$ and the 59 roots of unity aside from 1. Now, taking out the x^2 and doing a little bit of algebra on the second equation gives us:

   $$x^2(1 + x^2 + x^4 + x^6) = 0$$
   $$x^2\frac{x^8 - 1}{x^2 - 1} = 0$$
   $$x^2\frac{x^8 - 1}{(x - 1)(x + 1)} = 0$$

   From this we see that the solutions to the second equation are 0 and the 8th roots of unity excluding 1 and -1. Since the 8th roots of unity and the 59th roots of unity clearly do not overlap, our only overlapping solution is 0, and thus we have that $\boxed{58}$ is our answer.

   In general, if you see such a geometric progression, do not hesitate to take it out like we did here. Oftentimes, this will be a key to solving the problem!

   Finally, here's a harder one.

3. (AIME I 2018) Let N be the number of complex numbers z with the properties that $|z| = 1$ and $z^{6!} - z^{5!}$ is a real number. Find the remainder when N is divided by 1000.

   First of all, it's probably going to be easier for us to use $a = z^{120}$ (as then we also have $z^{720} = a^6$. Since for every z there is only one value of z^{120}, we know that there are exactly 120 unique solutions for z for every nonzero unique value of a, so after we solve for a our answer will just be what we got multiplied by 720.

   Now, having $a^6 - a$ be a real number is basically saying that the "imaginary parts" of a^6 and a are equal. This says that we either have $a^6 = a$, or a^6 and a are reflections across the imaginary axis. Note that the first condition translates to $(a^5 - 1)a = 0$, and so since $|a| = 1$ we have that all 5 of the 5th roots of unity.

   Now, we consider the second. Let $a = e^{i\theta}$ for some angle θ. We have that if a and a^6 are reflections across the imaginary axis, then their angles add up to an odd multiple of pi, and so we have (for some integer k):

$$\begin{aligned} 6\theta + \theta &= (2k-1)\pi \\ 7\theta &= (2k-1)\pi \\ \theta &= \frac{2k-1}{7}\pi \end{aligned}$$

   This gives 7 unique solutions for θ between 0 and 2π, and clearly these solutions do not intersect with the 5 roots of unity. Thus, we have that there are 12 possibilities for a, and $12 \cdot 120 = 1\boxed{440}$ solutions for z.

6 Problems

1. Let $w = 1 + 5i$ and $z = 6\sqrt{2} - 4i$. Compute the following:

   (a) $w + z$

   (b) $w - 2i$

   (c) w^3

   (d) $w^2 - z^2$

   (e) $wz - wz^2$

2. Find the number of ordered pairs of real numbers (a, b) such that $(a + bi)^{2002} = a - bi$.

   (A) 1001 (B) 1002 (C) 2001 (D) 2002 (E) 2004

3. The polynomial $f(x) = x^4 + ax^3 + bx^2 + cx + d$ has real coefficients, and $f(2i) = f(2 + i) = 0$. What is $a + b + c + d$?

   (A) 0 (B) 1 (C) 4 (D) 9 (E) 16

4. There is a complex number z with imaginary part 164 and a positive integer n such that

$$\frac{z}{z+n} = 4i.$$

   Find n.

5. Let

$$z = \frac{1+i}{\sqrt{2}}.$$

What is

$$\left(z^{1^2}+z^{2^2}+z^{3^2}+\cdots+z^{12^2}\right)\cdot\left(\frac{1}{z^{1^2}}+\frac{1}{z^{2^2}}+\frac{1}{z^{3^2}}+\cdots+\frac{1}{z^{12^2}}\right)?$$

(A) 18 **(B)** $72-36\sqrt{2}$ **(C)** 36 **(D)** 72 **(E)** $72+36\sqrt{2}$

6. Compute $(i+1)^3(i-2)^3+3(i+1)^2(i+3)(i-2)^2+3(i+1)(i+3)^2(i-2)+(i_1)^3$.

7. For certain real values of a, b, c, and d, the equation $x^4+ax^3+bx^2+cx+d=0$ has four non-real roots. The product of two of these roots is $13+i$ and the sum of the other two roots is $3+4i$, where $i=\sqrt{-1}$. Find b.

8. The points $(0,0)$, $(a,11)$, and $(b,37)$ are the vertices of an equilateral triangle. Find the value of ab.

7 Answers

1. just do it
2. e
3. d
4. 697
5. c
6. $-24i - 20$
7. 51
8. 315