Logarithms

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1 Introduction

This handout is designed to be an introduction to the concept of logarithms, and their use in AMC 10-12 level contest math. This handout is adapted from a tutoring series I did in the summer of 2021 (dm me for those handouts), with help from Eric Shen and AOPS Volume 2.

2 Key Identities

Definition: Logarithm notation is essentially the inverse of exponentation. Instead of writing

$$x = b^a$$

we can write

$$\log_b x = x$$

which is pronounced as "log base-b of x is equal to a"

Theorem: **Log Properties:** Given 3 numbers a, b, and x such that $\log_a x = b$, we have:

1. $\log_b(a) + \log_b(c) = \log_b(a \cdot c)$
2. $\log_b(a) - \log_b(c) = \log_b(\frac{a}{c})$
3. $\log_b(a^n) = n \cdot \log_b(a)$
4. $\log_{a^b}(c) = \frac{1}{b} \cdot log_a(c)$
5. $a^{\log_a(b)} = b$
6. $\log_b(a) = \frac{\log_c(a)}{\log_c(b)}$
7. $\log_a(b)\log_b(a) = 1$

Since the main focus of these handouts is for contest prep anyway, I figured it would be more helpful to put the formulas at the top :)

If you were to memorize these formulas for contest, I would suggest knowing numbers (1), (2), (3), (6), and (7), the formulas for addition, subtraction, exponents, change of base, and chain rule respectively. As such, let's try deriving a few of these properties ourselves:

First of all, probably the most basic idea in exponentation is that $a^b \cdot a^c = a^{b+c}$. Now, let's consider $x = a^b$, and $y = a^c$.

We have that $\log_a x = b$, and that $\log_a y = c$. We have that $\log_a xy = \log_a a^{b+c} = b + c$. Thus, we have a rule for converting sum to product:

$$\log_a x + \log_a y = \log_a xy$$

Similarly, we can do the same for differences. We know that $a^b \div a^c = a^{b-c}$, so we now have a rule for converting differences to quotients:

$$\log_a x - \log_a y = \log_a \frac{x}{y}$$

Okay, that's something to work off of. We also know that $(a^b)^c = a^{bc}$. If we say $x = a^b$, then we now have that:

$$\log_a x = b$$
$$\log_a x^c = bc = c \log_a x$$

Okay, that's great news. This basically should tell us how logarithms operate under the same base. But what about under different bases?

Well, let's think about it. Let's say we want to compare the two numbers $\log_a x = c$ and $\log_b x = d$. Then we have that:

$$a^c = x = b^d$$
$$a^{\frac{c}{d}} = b$$
$$\frac{c}{d} = \log_a b$$

Whoa! This now gives us:

$$c = d \log_a b$$
$$\log_a x = \log_a b \log_b x$$

This is known as the **chain rule** of logarithms. This, along with the above two rules, basically tell us all about how logarithms work.

3 Logarithm Algebra

Logarithm problems usually involve two main techniques:

1. Working with the three above identities.
2. Switching between logarithms and exponents.

3.1 Working with Logarithmic Identities

When we work with logarithmic identities, or whenever we work with a set of equations, it's always good to write down all of your equations/identities on the page. Personally, I find this helps because I can clearly see all the things I am working on at a given moment, so I don't forget to use anything.

A few heuristics:

1. Try to get everything to be roughly the same base, if this helps. This lets you use the addition property.
2. Substitute! A lot of the time logarithm questions are just quadratics / equations disguised by the fact that the variable *is* the logarithm, so subbing is always good!

Example 1 (AMC 2017)

How many ordered pairs (a, b) such that a is a real positive number and b is an integer between 2 and 200, inclusive, satisfy the equation $(\log_b a)^{2017} = \log_b(a^{2017})$?

Example 2 (AMC 2020)

Which of the following is the value of $\sqrt{\log_2 6 + \log_3 6}$?

Example 3 (AIME 2020)

There is a unique positive real number x such that the three numbers $\log_8(2x), \log_4 x$, and $\log_2 x$, in that order, form a geometric progression with positive common ratio. The number x can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

3.2 Switching between logarithms and exponents

The key idea here is this. Whenever we have an equation, we can always take the **logarithm** and the **exponents** of the entire side (with the same base). For example, the statement:

$$x + 1 = 3$$

can be converted to:

$$2^{x+1} = 2^3$$

OR

$$\log_3 x + 1 = \log_3 3$$

Since this may not seem terribly useful to you right now, here's an example:

Example 4 (AMC 12, modified)

Let:

$$\log_{10}(x + y) = z$$
$$\log_{10}(x^2 + y^2) = z + 1$$

Compute xy in terms of z.

Here this "switching" is basically a substitution of what the logarithm is. Nevertheless, a lot of the time you will find that the logarithm is more of just a "smokescreen", and that the equations are just as nice when the logarithms / exponents are taken out. Because of this, it's always good to try subbing and "switching".

Another good heuristic that it's time to switch: there's not an easy way to represent all the stuff in the logs. For example, in the AIME problem we just worked with logs because log 2x and log x are similar. Here this is not the case, and so exponentiation is probably going to be helpful.

Our final problem!

Example 5 (AIME 2018)

For each ordered pair of real numbers (x, y) satisfying

$$\log_2(2x + y) = \log_4(x^2 + xy + 7y^2)$$

there is a real number K such that

$$\log_3(3x + y) = \log_9(3x^2 + 4xy + Ky^2).$$

Find the product of all possible values of K.

4 Exercises

1. (2) What is the value of $(625^{\log_5 2015})^{\frac{1}{4}}$?
2. (3) There is a unique positive integer n such that

$$\log_2(\log_{16} n) = \log_4(\log_4 n).$$

   What is the sum of the digits of n?
3. (3) When $p = \sum_{k=1}^{6} k \ln k$, the number e^p is an integer. What is the largest power of 2 that is a factor of e^p?
4. (3) The sequence

$$\log_{12} 162,\ \log_{12} x,\ \log_{12} y,\ \log_{12} z,\ \log_{12} 1250$$

   is an arithmetic progression. What is x?
5. (3) What is the value of

$$\log_3 7 \cdot \log_5 9 \cdot \log_7 11 \cdot \log_9 13 \cdots \log_{21} 25 \cdot \log_{23} 27?$$

6. (3) What is the value of a for which $\frac{1}{\log_2 a} + \frac{1}{\log_3 a} + \frac{1}{\log_4 a} = 1$?
7. (3) The value of x that satisfies $\log_{2^x} 3^{20} = \log_{2^{x+3}} 3^{2020}$ can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.
8. (3) The solution of the equation $7^{x+7} = 8^x$ can be expressed in the form $x = \log_b 7^7$. What is b?
9. (3) For what value of x does

$$\log_{\sqrt{2}} \sqrt{x} + \log_2 x + \log_4(x^2) + \log_8(x^3) + \log_{16}(x^4) = 40?$$

10. (4) Positive real numbers $x \neq 1$ and $y \neq 1$ satisfy $\log_2 x = \log_y 16$ and $xy = 64$. What is $(\log_2 \frac{x}{y})^2$?

11. (4) Positive real numbers a and b have the property that

$$\sqrt{\log a} + \sqrt{\log b} + \log\sqrt{a} + \log\sqrt{b} = 100$$

and all four terms on the left are positive integers, where log denotes the base 10 logarithm. What is ab?

12. (4) The graphs of $y = \log_3 x$, $y = \log_x 3$, $y = \log_{\frac{1}{3}} x$, and $y = \log_x \frac{1}{3}$ are plotted on the same set of axes. How many points in the plane with positive $x-$coordinates lie on two or more of the graphs?

13. (4) The vertices of a quadrilateral lie on the graph of $y = \ln x$, and the x-coordinates of these vertices are consecutive positive integers. The area of the quadrilateral is $\ln \frac{91}{90}$. What is the x-coordinate of the leftmost vertex?

14. (6) Let $a = 256$. Find the unique real number $x > a^2$ such that

$$\log_a \log_a \log_a x = \log_{a^2} \log_{a^2} \log_{a^2} x.$$

15. (6) Real numbers x and y are chosen independently and uniformly at random from the interval $(0, 1)$. What is the probability that $\lfloor \log_2 x \rfloor = \lfloor \log_2 y \rfloor$, where $\lfloor r \rfloor$ denotes the greatest integer less than or equal to the real number r?

16. (6) Let x, y and z be real numbers satisfying the system

$$\begin{aligned} \log_2(xyz - 3 + \log_5 x) &= 5 \\ \log_3(xyz - 3 + \log_5 y) &= 4 \\ \log_4(xyz - 3 + \log_5 z) &= 4. \end{aligned}$$

Find the value of $|\log_5 x| + |\log_5 y| + |\log_5 z|$.

17. (6) There are positive integers x and y that satisfy the system of equations

$$\begin{aligned} \log_{10} x + 2\log_{10}(\gcd(x, y)) &= 60 \\ \log_{10} y + 2\log_{10}(\operatorname{lcm}(x, y)) &= 570. \end{aligned}$$

Let m be the number of (not necessarily distinct) prime factors in the prime factorization of x, and let n be the number of (not necessarily distinct) prime factors in the prime factorization of y. Find $3m + 2n$.

18. (6) Find the number of integer values of k in the closed interval $[-500, 500]$ for which the equation $\log(kx) = 2\log(x + 2)$ has exactly one real solution.

19. (9) In a Martian civilization, all logarithms whose bases are not specified are assumed to be base b, for some fixed $b \geq 2$. A Martian student writes down

$$\begin{aligned} 3\log(\sqrt{x}\log x) &= 56 \\ \log_{\log(x)}(x) &= 54 \end{aligned}$$

and finds that this system of equations has a single real number solution $x > 1$. Find b.

20. (14) Let $a > 1$ and $x > 1$ satisfy $\log_a(\log_a(\log_a 2) + \log_a 24 - 128) = 128$ and $\log_a(\log_a x) = 256$. Find the remainder when x is divided by 1000.

21. (2) Find x such that $7^{\log_{14}(41)} = x^{\log_{14}(7)}$