Algebra

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1 Introduction

This handout is designed to be an introduction to AMC 10/12 level algebra. Most pre-Olympiad level polynomial algebra can be solved using a few key formulas, with the rest being intuition and manipulations. Also, thanks Eric Shen :)

2 Manipulations

Theorem: Common Algebraic Manipulations:

1. $(a \pm b)^2 = a^2 \pm 2ab + b^2$ (square of sum/difference)
2. $(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + ac + bc)$ (square of 3 sums)
3. $a^2 - b^2 = (a + b)(a - b)$ (difference of squares)
4. $(a \pm b)^3 = a^3 \pm 3a^2b + 3b^2a \pm b^3$ (cube of sum/difference)
5. $a^3 \pm b^3 = (a \pm b)(a^2 \pm ab + b^2)$ (sum/difference of cubes)
6. $a^3 + b^3 + c^3 = (a + b + c)(a^2 + b^2 + c^2 - ab - ac - bc)$ (cube of 3 sums)
7. $a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + \ldots + ab^{n-2} + bn - 1)$ (generalized difference)
8. $a^n + b^n = (a + b)(a^{n-1} + a^{n-2}b + \ldots + ab^{n-2} + bn - 1)$ (generalized sum for odd n)

3 Roots

Definition: Broadly speaking, a polynomial is the combination of more than one integer powers. The general form of a polynomial is:

$$P(x) = a_nx^n + a_{n-1} + \ldots a_0$$

Given a polynomial $P(x)$, any k such that $P(k) = 0$ is considered a root of P.

Consider the difference $P(x) - P(k) = a_n(x^n - k^n) + \ldots a_0 - a_0$. Since $P(k) = 0$, the difference still equals $P(x)$. In addition, note that since $a - b$ is always a factor of $a^n - b^n$, we have that $x - k$ is a factor of $a_i(x^i - k^i)$ for all $0 \leq i \leq n$. This means that since $P(x)$ is the sum of all such terms, $x - k$ is a factor of $P(x) - P(k) = P(x)$ as well.

Theorem: Factor Theorem:

Given a polynomial $P(x) = a_n x^n + a_{n-1} + \ldots a_0$, $(x - k)$ is a factor of $P(x)$ if and only if $P(k) = 0$, or if k is a root of P

This can be generalized for any k, regardless of whether it is a root or not. Note that since $x - k$ is a factor of $P(x) - P(k)$ regardless of k, we have that dividing $P(x)$ by $x - k$ yields a remainder of $P(k)$.

Theorem: Remainder Theorem:

Given a polynomial $P(x) = a_n x^n + a_{n-1} + \ldots a_0$, the remainder of $P(x)$ divided by any $x - k$ is $P(k)$

The last theorem worthy of mentioning is the Fundamental Theorem of Algebra:

Theorem: Fundamental Theorem of Algebra:

Given a polynomial $P(x)$ of the nth degree, $P(x)$ has exactly n complex roots, each of which can be expressed as $a + bi$.

This theorem appears much less on contests, however its largest application is being able to write polynomials in factored form:

Definition: Given the n roots $x_1, x_2, \ldots x_n$ of a polynomial $P(x)$, we have that

$$P(x) = a(x - x_1)(x - x_2)\ldots(x - x_n)$$

4 Rational Root Theorem

This is essentially an extension of the Polynomial Remainder Theorem. Given a polynomial $P(x)$ in standard form with k as a root, we have that

$$a_n k^n + \ldots a_1 k + a_0 = 0$$

$$k(a_n k^{n-1} + \ldots a_1) = -a_0$$

This implies that k divides a_0, but it can be generalized to all rational numbers. Let $k = \frac{p}{q}$, where p and q are coprime. We now have that:

$$a_n(\frac{p}{q})^n + \cdots a_1\frac{p}{q} + a_0 = 0$$

$$a_n p^n + a_{n-1}p^{n-1}q + \ldots + a_0 q^n = 0$$

We now have that q divides $a_0 q^n$ and that p divides a_0, giving us:

Theorem: Rational Root Theorem:

Given a polynomial $P(x)$, all rational solutions of $P(x)$ can be written as $\frac{p}{q}$, where p and q are coprime integers, p divides the last term a_0, and q divides the first term $a_n x^n$.

This theorem helps us factor polynomials of degree ≥ 2, without using substitution.

4.1 Examples

Example 1 (Basic RRT)

Find all roots of the polynomial $2x^3 + 5x^2 + x - 2$

Example 2 (Remainder Theorem)

A quartic polynomial $ax^4 + bx^3 + cx + d$ has roots $1+\sqrt{3}, 1-\sqrt{3}, 2+\sqrt{2}$, and $2-\sqrt{2}$. Calculate $a+b+c+d$.

5 Vieta's Formulas

Given a quadratic equation $ax^2 + bx + c = 0$, you probably already know that the two values of x are $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$. However, we want to know how these two roots (p, q) relate, namely what their sum and product are. Let's start with the sum:

$$p + q = \frac{-b+\sqrt{b^2-4ac}}{2a} + \frac{-b-\sqrt{b^2-4ac}}{2a}$$

$$p + q = \frac{-2b}{2a}$$

$$p + q = -\frac{b}{a}$$

Then, the product:

$$pq = (\frac{-b+\sqrt{b^2-4ac}}{2a})(\frac{-b-\sqrt{b^2-4ac}}{2a})$$

The numerator can be expressed as a difference of squares:

$$pq = \frac{b^2 - b^2 + 4ac}{(2a)^2}$$

$$pq = \frac{4ac}{4a^2}$$

$$pq = \frac{c}{a}$$

Now, let's try the same with a cubic, however given that the cubic formula is much more complex, we'll use its factored form:

$$a(x-p)(x-q)(x-r) = ax^3 + bx^2 + cx + d$$

$$ax^3 - a(p+q+r)x^2 + a(pq+qr+rp)x - apqr = ax^3 + bx^2 + cx + d$$

Comparing coefficients gives us

$$p + q + r = -\frac{b}{a}$$

$$pq + qr + p = \frac{c}{a}$$

$$pqr = -\frac{d}{a}$$

These formulas generalize nicely, and are known as Vieta's Formulas. Make sure you understand how these work, as they are the foundation of many AMC algebra problems.

Theorem: Vieta's Theorems:

Given a polynomial $P(x) = a_n x^n + a_{n-1} + \dots a_0$ with n (not necessarily distinct) complex roots, we have that

$$r_1 + r_2 + \cdots + r_n = -\frac{a_{n-1}}{a_n}$$

$$r_1 r_2 + r_1 r_3 + \cdots + r_{n-1} r_n = \frac{a_{n-2}}{a_n}$$

$$\vdots$$

$$r_1 r_2 r_3 \cdots r_n = (-1)^n \frac{a_0}{a_n}.$$

Compactly, this equates to

$$\sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} \left(\prod_{j=1}^{k} r_{i_j} \right) = (-1)^k \frac{a_{n-k}}{a_n}$$

5.1 Examples

Example 3 (Basic Vietas)

Let $g(x) = x^3 - 4x^2 + 5x - 8$, and let its roots be p, q, and r. Compute $p^2qr + pq^2r + pqr^2$.

Example 4 (David Altizio)

The polynomial $x^3 - Ax + 15$ has three real roots. Two of these roots sum to 5. What is $|A|$?

Example 5 (USAMO 1984)

The product of two of the four roots of the quartic equation $x^4 - 18x^3 + kx^2 + 200x - 1984$ is -32. Determine the value of k.

6 Problems

1. Completely factor the polynomial $x^4 - x^3 - 5x^2 + 3x + 6$

2. Compute the sum of all the roots of $(2x+3)(x-4)+(2x+3)(x-6)=0$

   (A) $\frac{7}{2}$ (B) 4 (C) 5 (D) 7 (E) 13

3. Find the remainder when x^{13} is divided by $x-1$.

4. Suppose that a and b are nonzero real numbers, and that the equation $x^2+ax+b=0$ has solutions a and b. Then the pair (a,b) is

   (A) $(-2,1)$ (B) $(-1,2)$ (C) $(1,-2)$ (D) $(2,-1)$ (E) $(4,4)$

5. What is the sum of the reciprocals of the roots of the equation $\frac{2003}{2004}x + 1 + \frac{1}{x} = 0$?

   (A) $-\frac{2004}{2003}$ (B) -1 (C) $\frac{2003}{2004}$ (D) 1 (E) $\frac{2004}{2003}$

6. Let $f(x) = ax^7 + bx^3 + cx - 5$, where a, b and c are constants. If $f(-7) = 7$, then $f(7)$ equals

   (A) -17 (B) -7 (C)14 (D)21 (E)not uniquely determined

7. Let a and b be the roots of the equation $x^2 - mx + 2 = 0$. Suppose that $a + \frac{1}{b}$ and $b + \frac{1}{a}$ are the roots of the equation $x^2 - px + q = 0$. What is q?

   (A) $\frac{5}{2}$ (B) $\frac{7}{2}$ (C) 4 (D) $\frac{9}{2}$ (E) 8

8. If r_1, r_2, r_3 are the three distinct solutions to the equation $2x^3 + x + 9 = 0$, what is the value of $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$?

9. Let Q be a polynomial

   $$Q(x) = a_0 + a_1 x + \cdots + a_n x^n,$$

   where $a_0, \ldots, a_n$ are nonnegative integers. Given that $Q(1) = 4$ and $Q(5) = 152$, find $Q(6)$.

10. The polynomial $f(x) = x^4 + ax^3 + bx^2 + cx + d$ has real coefficients, and $f(2i) = f(2+i) = 0$. What is $a+b+c+d$?

    (A) 0 (B) 1 (C) 4 (D) 9 (E) 16

11. Consider all lines that meet the graph of $y = 2x^4 + 7x^3 + 3x - 5$ in four distinct points: (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4). Show that

    $$\frac{x_1 + x_2 + x_3 + x_4}{4}$$

    is independent of the line and find its value.

12. The function $f(x) = x^4 + ax^3 + bx^2 + 2000x + d$ has four distinct roots. Two of the rots sum to 5; the other two roots also sum to 5. Compute b.