AMC 12B 2022 Solutions

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1 Credits

Problems and some solutions courtesy of cool people on AOPS, you should check them out - here's a list of the problems and links to the solutions: https://artofproblemsolving.com/community/c5h2962544

2 Solutions

1. Define $x \diamond y$ to be $|x-y|$ for all real numbers x and y. What is the value of

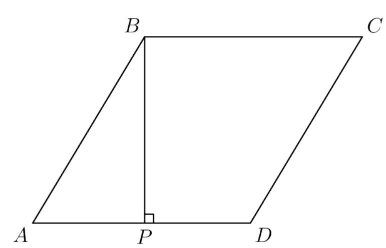
$$(1 \diamond (2 \diamond 3)) - ((1 \diamond 2) \diamond 3)?$$

(A) -2 **(B)** -1 **(C)** 0 **(D)** 1 **(E)** 2

Solution:

$$\begin{aligned} &= |1-|2-3|| - ||1-2|-3| \\ &= |1-1| - |1-3| \\ &= \boxed{\textbf{(A)}\ -2} \end{aligned}$$

2. In rhombus $ABCD$, point P lies on segment $\overline{AD}$ such that $BP \perp AD$, $AP = 3$, and $PD = 2$. What is the area of $ABCD$?



(A) $3\sqrt{5}$ **(B)** 10 **(C)** $6\sqrt{5}$ **(D)** 20 **(E)** 25

Solution: Since a rhombus has sides with equal side lengths, we know $\overline{AB} = 1$. By Pythagoras, $\overline{BP} = 4$. Finally, since the area of a rhombus is equal to base times height, the answer is $\boxed{\textbf{(D)}\ 20}$

3. How many of the first ten numbers of the sequence 121, 11211, 1112111, ... are prime numbers?

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

Solution: This was a very easy question to silly on the test - my instinct was to cancel using divisibility rules for 3 and 11. However, there exists the following construction:

$$121 = 11 + 110 = 11(1 + 10)$$

$$11211 = 111 + 11100 = 11(1 + 10^2)$$
$$1112111 = 1111 + 1111000 = 11(1 + 10^3)$$
$$\cdots$$

Thus, all numbers of that form are composite, and our answer is $\boxed{\text{(A) } 0}$

4. For how many values of the constant k will the polynomial $x^2+kx+36$ have two distinct integer roots?

   (A) 6 **(B)** 8 **(C)** 9 **(D)** 14 **(E)** 16

   Solution: This is easily solved when thinking of sum-product factoring - we need two integers that multiply to 36 and sum to k.

   $36 = 1 \cdot 36 = 2 \cdot 18 = 3 \cdot 12 = 4 \cdot 9$

   Note that $(6, 6)$ doesn't work because it would lead to a perfect square quadratic with one solution. Also note that taking the negative of the above pairs also works, so our solution is $4 \cdot 2 = \boxed{\textbf{(B) } 8}$

5. The point $(-1, -2)$ is rotated $270°$ counterclockwise about the point $(3, 1)$. What are the coordinates of its new position?

   (A) $(-3, -4)$ **(B)** $(0, 5)$ **(C)** $(2, -1)$ **(D)** $(4, 3)$ **(E)** $(6, -3)$

   Solution: You can do this any way that you want - I just drew it out, which gives you the answer $\boxed{\textbf{(B) } (0, 5)}$

6. Consider the following 100 sets of 10 elements each:

$$\{1, 2, 3, \cdots, 10\},$$
$$\{11, 12, 13, \cdots, 20\},$$
$$\{21, 22, 23, \cdots, 30\},$$
$$\vdots$$
$$\{991, 992, 993, \cdots, 1000\}.$$

   How many of these sets contain exactly two multiples of 7?

   (A)40 **(B)**42 **(C)**42 **(D)**49 **(E)**50

   Solution: We try to write all these numbers modulo 7. Note that we only need to write out the first and last number of each row, and that we can stop after 7 rows (as everything repeats):

$$1 \cdots 10 \equiv 1 \cdots 3 (\text{mod} 7)$$
$$11 \cdots 20 \equiv 4 \cdots 6 (\text{mod} 7)$$
$$21 \cdots 30 \equiv 0 \cdots 2 (\text{mod} 7)$$
$$31 \cdots 40 \equiv 3 \cdots 5 (\text{mod} 7)$$
$$41 \cdots 50 \equiv 6 \cdots 1 (\text{mod} 7)$$
$$51 \cdots 60 \equiv 2 \cdots 4 (\text{mod} 7)$$
$$61 \cdots 70 \equiv 5 \cdots 0 (\text{mod} 7)$$

   Note that sets 3, 5, and 7 will always contain two multiples of 7 (0 mod 7). Thus, every seven sets, three of them will contain two multiples of 7. $14 \cdot 70 = 980$, so all that remains is to check $981 - 1000$. This is equivilent to the first two rows, which each only contain one multiple of 7. Thus, our answer is $3 \cdot 14 = \boxed{\textbf{(B)} 42}$

7. Camila writes down five positive integers. The unique mode of these integers is 2 greater than their median, and the median is 2 greater than their arithmetic mean. What is the least possible value for the mode?

(A) 5 **(B)** 7 **(C)** 9 **(D)** 11 **(E)** 13

Solution: Letting the median be m, we can write the set as $\{a, b, m, m+2, m+2\}$, where a and b must satisfy $m = 2 + \frac{a+b+3m+2}{5}$. On contest, it would be faster to just bash out the answers from smallest to largest. Just one note of caution- you might think that $\{1, 1, 7, 9, 9\}$ satisfies C, but the problem statement says that there must be a unique mode. Thus, our answer is $\boxed{\textbf{(D) } 11}$

8. What is the graph of $y^4 + 1 = x^4 + 2y^2$ in the coordinate plane?

(A) Two intersecting parabolas **(B)** Two nonintersecting parabolas **(C)** Two intersecting circles **(D)** A circle and a hyperbola **(E)** A circle and two parabolas

Solution: Rearrange and factor:

$$y^4 - 2y^2 - 1 = x^4$$
$$(y^2 - 1)^2 - x^4 = 0$$
$$(y^2 + x^2 - 1)(y^2 - x^2 - 1) = 0$$

One of these must equal 0 - if the left side does then the graph is a hyperbola, and if the right side does then the graph is a circle of radius 1. Thus, our answer is $\boxed{\textbf{(D)}}$

9. The sequence $a_0, a_1, a_2, \cdots$ is a strictly increasing arithmetic sequence of positive integers such that

$$2^{a_7} = 2^{27} \cdot a_7.$$

What is the minimum possible value of a_2?

(A)8 **(B)**12 **(C)**16 **(D)**17 **(E)**22

Solution: Let $a^7 = 2^n$:

$$2^{2^x} = 2^{27+x}$$
$$2^x = 27 + x \implies x = 5 \implies a_7 = 32$$

Now given that it's an arithmetic sequence, we let $a_7 = a_0 + 7d$. We want to minimize $a_0 + 2d$, which happens when $a_0 = d = 4$. Thus, our answer is $4 + 2(4) = \boxed{\textbf{(B)}12}$

10. Regular hexagon $ABCDEF$ has side length 2. Let G be the midpoint of $\overline{AB}$, and let H be the midpoint of $\overline{DE}$. What is the perimeter of $GCHF$? **(A)** $4\sqrt{3}$ **(B)** 8 **(C)** $4\sqrt{5}$ **(D)** $4\sqrt{7}$ **(E)** 12

Solution: The perimeter of $GCHF$ is equal to $4 \cdot \overline{CH}$. $\overline{CH}$ is a hypotenuse with legs 2 and $\sqrt{3}$.

$$\overline{CH} = \sqrt{2^2 + \sqrt{3}^2} = \sqrt{7}$$

Thus, our answer is $4 \cdot \sqrt{7} = \boxed{\textbf{(D) } 4\sqrt{7}}$.

11. Let $f(n) = \left(\frac{-1+i\sqrt{3}}{2}\right)^n + \left(\frac{-1-i\sqrt{3}}{2}\right)^n$, where $i = \sqrt{-1}$. What is $f(2022)$

(A) -2 **(B)** -1 **(C)** 0 **(D)** $\sqrt{3}$ **(E)** 2

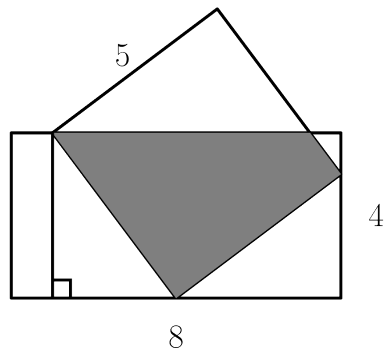
Solution: Either bashing the first 4 iterations of n and discovering a pattern or using De Moivre's yields $\boxed{\textbf{(E) } 2}$

12. Kayla rolls four fair 6-sided dice. What is the probability that at least one of the numbers Kayla rolls is greater than 4 and at least two of the numbers she rolls are greater than 2?

(A)$\frac{2}{3}$ **(B)**$\frac{19}{27}$ **(C)**$\frac{59}{81}$ **(D)**$\frac{61}{81}$ **(E)**$\frac{7}{9}$

Solution: $\boxed{\textbf{(D)}\frac{61}{81}}$

13. The diagram below shows a rectangle with side lengths 4 and 8 and a square with side length 5. Three vertices of the square lie on three different sides of the rectangle, as shown. What is the area of the region inside both the square and the rectangle?



(A)$15\frac{1}{8}$ **(B)**$15\frac{3}{8}$ **(C)**$15\frac{1}{2}$ **(D)**$15\frac{5}{8}$ **(E)**$15\frac{7}{8}$

Solution: Using Pythagoras and similiar triangles, the two large right-angled triangles are both 3-4-5 triangles. Then the height of the small triangle is 1, and we set it's base to be x. Using similar triangles, we have:

$$\frac{x}{1} = \frac{5-\sqrt{x^2+1}}{5}$$
$$5x = 5-\sqrt{x^2+1}$$
$$(5x-5)^2 = x^2+1$$
$$25x^2-50x+25 = x^2+1$$
$$24x^2-50x+24 = 0$$
$$(3x-4)(8x-6) = 0$$

Thus, the height is $\frac{3}{4}$. The area of the shaded region is then equal to

$$7 \cdot 4 - 2(\frac{3\dot{4}}{2}) - \frac{1 \cdot \frac{3}{4}}{2} = \boxed{\textbf{(D)}15\frac{5}{8}}$$

14. The graph of $y = x^2+2x-15$ intersects the x-axis at points A and C and the y-axis at point B. What is $\tan(\angle ABC)$?

(A)$\frac{1}{7}$ **(B)**$\frac{1}{4}$ **(C)**$\frac{3}{7}$ **(D)**$\frac{1}{2}$ **(E)**$\frac{4}{7}$

Solution: $y = (x+5)(x-3)$ gives us that the x-intercepts are -5 and 3, and that the y-intercept is -15. This gives us two right-angled triangles with legs $(5, 15)$ and $(3, 15)$. Since tangent is equal to opposite over adjacent, the tangent of each of the angles is $\frac{1}{3}$ and $\frac{1}{5}$. To find the tangent of the entire angle, we use the sum of tangents formula:

$$\tan(x+y) = \frac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)}$$
$$\frac{1}{3}+\frac{1}{5} = \frac{\frac{1}{3}+\frac{1}{5}}{1-(\frac{1}{3})(\frac{1}{5})}$$
$$\frac{\frac{8}{15}}{\frac{14}{15}} = \frac{8}{14} = \frac{4}{7}$$

Thus, our answer is $\boxed{\textbf{(E)}\frac{4}{7}}$

15. One of the following numbers is not divisible by any prime number less than 10. Which is it? (A) $2^{606}-1$ (B) $2^{606}+1$ (C) $2^{607}-1$ (D) $2^{607}+1$ (E) $2^{607}+3^{607}$

    Solution: All powers of 2 have patterns modulo 3 and 5:

    powers of 2 mod 3:

$$
\begin{aligned}
2 &\equiv 2 \\
4 &\equiv 1
\end{aligned}
$$

    powers of 2 mod 5:

$$
\begin{aligned}
2 &\equiv 2 \\
4 &\equiv 4 \\
8 &\equiv 3 \\
16 &\equiv 1
\end{aligned}
$$

    As such, we can just eliminate answer choices:

    a) $1-1=0$ mod 3
    b) $4+1=0$ mod 5
    c) Merrense prime
    d) $2+1=0$ mod 3
    e) 0 mod 5

    Thus, our answer is $\boxed{\textbf{(C)}2^{607}-1}$

16. Suppose x and y are positive real numbers such that

    $x^y = 2^{64}$ and $(\log_2 x)^{\log_2 y} = 2^{27}$.

    What is the greatest possible value of $\log_2 y$?

    $\textbf{(A)}3$ $\textbf{(B)}4$ $\textbf{(C)}3+\sqrt{2}$ $\textbf{(D)}4+\sqrt{3}$ $\textbf{(E)}7$

    Solution: Let $\log_2 x = a, \log_2 y = b$. Then the equations become $a^b = 2^7$ and $2^{a\cdot 2^b} = 2^{64} \implies a\cdot 2^b = 64 = 2^6 \implies a = 2^{6-b}$. Substituting, $2^{b(6-b)} = 2^7$, so $b(6-b) = 7$ which is a quadratic that has largest root $\boxed{\textbf{(C)}3+\sqrt{2}}$

17. How many 4×4 arrays whose entries are 0s and 1s are there such that the row sums (the sum of the entries in each row) are $1, 2, 3,$ and 4, in some order, and the column sums (the sum of the entries in each column) are also $1, 2, 3,$ and 4, in some order? For example, the array

$$
\begin{bmatrix}
1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0
\end{bmatrix}
$$

    satisfies the condition.

    $\textbf{(A)}144$ $\textbf{(B)}240$ $\textbf{(C)}336$ $\textbf{(D)}576$ $\textbf{(E)}624$

    Solution: $4\cdot 4$ choices for a row and column of 1s.

    $3\cdot 3$ choices for a row and column consisting of 3 0s and 1 1.

    4 choices for a final 0 and we fill the rest up with 1s.

    $4\cdot 4\cdot 3\cdot 3\cdot 4 = \boxed{\textbf{(D)}576}$

18. Each square in a 5×5 grid is either filled or empty, and has up to eight adjacent neighboring squares, where neighboring squares share either a side or a corner. The grid is transformed by the following rules: Any filled square with two or three filled neighbors remains filled. Any empty square with exactly three filled neighbors becomes a filled square. All other squares remain empty or become empty.

    Suppose the 5×5 grid has a border of empty squares surrounding a 3×3 subgrid. How many initial configurations will lead to a transformed grid consisting of a single filled square in the center after a single transformation? (Rotations and reflections of the same configuration are considered different.)

    (A) **14** **(B)** **18** **(C)** **22** **(D)** **26** **(E)** **30**

    Solution: Bashing yields $\boxed{\textbf{(C) } \mathbf{22}}$

19. In $\triangle ABC$ medians $\overline{AD}$ and $\overline{BE}$ intersect at G and $\triangle AGE$ is equilateral. Then $\cos(C)$ can be written as $\frac{m\sqrt{p}}{n}$, where m and n are relatively prime positive integers and p is a positive integer not divisible by the square of any prime. What is $m+n+p$?

    (A)44 **(B)**48 **(C)**52 **(D)**56 **(E)**60

    Solution: Let the foot from D to AC be F. Note that

    $$\frac{\sqrt{3}}{4}AE^2 = [AGE] = \frac{1}{6}[ABC] = \frac{2}{3}[ADC] = \frac{1}{3} \cdot AC \cdot DF = \frac{2}{3} \cdot AE \cdot DF$$

    hence $DF = \frac{3\sqrt{3}}{4}AE$. Thus, we know that $GD = \frac{1}{2}AE$, so thus

    $$AF = \sqrt{AD^2 - DF^2} = AE\sqrt{\frac{9}{4} - \frac{27}{16}} = \frac{3}{4}AE$$

    Thus, $CF = \frac{5}{4}AE$, and by the Pythagorean Theorem again,

    $$CD = \sqrt{DF^2 + FC^2} = AE\sqrt{\frac{2}{5}16 + \frac{27}{1}6} = \frac{\sqrt{13}}{2}$$

    In particular, we have

    $$\cos\angle C = \frac{CF}{CD} = \frac{5\sqrt{13}}{26}$$

    so the answer is $5 + 13 + 26 = \boxed{\textbf{(A)}44}$.

20. Let $P(x)$ be a polynomial with rational coefficients such that when $P(x)$ is divided by the polynomial $x^2 + x + 1$, the remainder is $x + 2$, and when $P(x)$ is divided by the polynomial $x^2 + 1$, the remainder is $2x + 1$. There is a unique polynomial of least degree with these two properties. What is the sum of the squares of the coefficients of that polynomial?

    (A) 10 **(B)** 13 **(C)** 19 **(D)** 20 **(E)** 23

    Solution: Say $P(x) = ax^3 + bx^2 + cx + d$
    Dividing $P(x)$ by $x^2 + 1$, the remainder is $(c-a)x + (d-b)$
    Dividing $P(x)$ by $x^2 + x + 1$, the remainder is $(c-b)x + (d-b+a)$
    Thus:

    $$c - a = 2$$

    $$d - b = 1$$

    $$c - b = 1$$

    $$d - b + a = 2$$

    Thus, a=1, giving c=3, b=2, and d=3. The answer is thus $1^2 + 2^2 + 3^2 + 3^2 = \boxed{\textbf{(E) } 23}$

21. Let S be the set of circles in the coordinate plane that are tangent to each of the three circles with equations $x^2 + y^2 = 4$, $x^2 + y^2 = 64$, and $(x-5)^2 + y^2 = 3$. What is the sum of the areas of all circles in S?

    (A) 48π **(B)** 68π **(C)** 96π **(D)** 102π **(E)** 136π

    Solution: $\boxed{\textbf{(E)}136}$

22. Ant Amelia starts on the number line at 0 and crawls in the following manner. For $n = 1, 2, 3$, Amelia chooses a time duration t_n and an increment x_n independently and uniformly at random from the interval $(0, 1)$. During the nth step of the process, Amelia moves x_n units in the positive direction, using up t_n minutes. If the total elapsed time has exceeded 1 minute during the nth step, she stops at the end of that step; otherwise, she continues with the next step, taking at most 3 steps in all. What is the probability that Amelia's position when she stops will be greater than 1?

    (A) $\frac{1}{3}$ **(B)** $\frac{1}{2}$ **(C)** $\frac{2}{3}$ **(D)** $\frac{3}{4}$ **(E)** $\frac{5}{6}$

    Solution: $\boxed{\textbf{(C)}\ \frac{2}{3}}$

23. Let $x_0, x_1, x_2, \cdots$ be a sequence of numbers, where each x_k is either 0 or 1. For each positive integer n, define

$$S_n = \sum_{k=0}^{n-1} x_k 2^k$$

    Suppose $7S_n \equiv 1 \pmod{2^n}$ for all $n \geq 1$. What is the value of the sum

$$x_{2019} + 2x_{2020} + 4x_{2021} + 8x_{2022}?$$

    (A) 6 **(B)** 7 **(C)** 12 **(D)** 14 **(E)** 15

    Solution: $\boxed{\textbf{(A)}6}$

24. The figure below depicts a regular 7-gon inscribed in a unit circle. What is the sum of the 4th powers of the lengths of all 21 of its edges and diagonals?

    (A)49 **(B)**98 **(C)**147 **(D)**168 **(E)**196

    Solution: $\boxed{\textbf{(C)}147}$

25. Four regular hexagons surround a square with a side length 1, each one sharing an edge with the square, as shown in the figure below. The area of the resulting 12-sided outer nonconvex polygon can be written as $m\sqrt{n} + p$, where m, n, and p are integers and n is not divisible by the square of any prime. What is $m + n + p$?

    (A) -12 **(B)** -4 **(C)**4 **(D)**24 **(E)**32

    Solution: $\boxed{\textbf{(B)}-4}$