AMC 12A 2022 Solutions

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1 Credits

Problems and some solutions courtesy of cool people on AOPS, you should check them out - here's a list of the problems and links to the solutions: https://artofproblemsolving.com/community/c5h2958350

2 Solutions

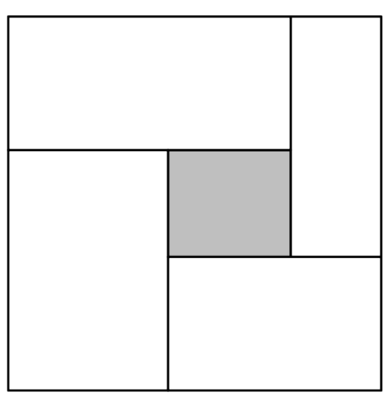
1. What is the value of
$$3+\frac{1}{3+\frac{1}{3+\frac{1}{3}}}?$$
(A) $\frac{31}{10}$ **(B)** $\frac{49}{15}$ **(C)** $\frac{33}{10}$ **(D)** $\frac{109}{33}$ **(E)** $\frac{15}{4}$

   Solution: Pure computation yields $\boxed{\textbf{(D)}\frac{109}{33}}$

2. The sum of three numbers is 96. The first number is 6 times the third number, and the third number is 40 less than the second number. What is the absolute value of the difference between the first and second numbers? **(A)** 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

   Solution: Set the numbers to be $(6x, x+40, x)$, giving us $8x+40=96 \implies x=7$. This gives us $47-42=\boxed{\textbf{(E)}\ 5}$

3. Five rectangles, A, B, C, D, and E, are arranged in a square as shown below. These rectangles have dimensions 1×6, 2×4, 5×6, 2×7, and 2×3, respectively. (The figure is not drawn to scale.) Which of the five rectangles is the shaded one in the middle?



(A) A **(B)** B **(C)** C **(D)** D **(E)** E

Solution: The sum of the areas of all rectangles is 64, so the side length of the square is 8.

Now the 2×4 rectangle can't be on the edge, as it's the only rectangle with side length 4. Answer $\boxed{\textbf{(B)}}$

TAKEAWAY: Sometimes, bashing just works

4. The least common multiple of a positive integer n and 18 is 180, and the greatest common divisor of n and 45 is 15. What is the sum of the digits of n?

   (A) 3 **(B)** 6 **(C)** 8 **(D)** 9 **(E)** 12

   Solution: Trial and error yields $n = 60$, so the sum of digits is $\boxed{\textbf{(B) } 6}$

5. Let the taxicab distance between points (x_1, y_1) and (x_2, y_2) in the coordinate plane is given by $|x_1 - x_2| + |y_1 - y_2|$. For how many points P with integer coordinates is the taxicab distance between P and the origin less than or equal to 20?

   (A) 441 **(B)** 761 **(C)** 841 **(D)** 921 **(E)** 924

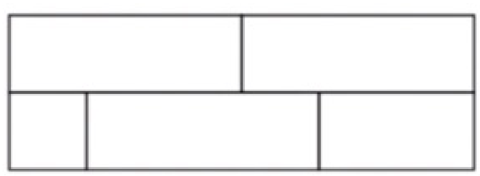
   Solution: $|x| + |y| \leq 20$, $1 + 3 + \cdots + 39 + 41 + 39 + \cdots + 3 + 1 = 21^2 + 20^2 = \boxed{\textbf{(C) } 841}$

   TAKEAWAY: Reading the question is important

6. A data set consists of 6 (not distinct) positive integers: 1, 7, 5, 2, 5, and X. The average (arithmetic mean) of the 6 numbers equals a value in the data set. What is the sum of all positive values of X?
   (A) 10 **(B)** 26 **(C)** 32 **(D)** 36 **(E)** 40

   Solution: Testing different intervals gives us $x = 4$ (where the average is 4), $x = 10$ (where the average is 5), and $x = 22$ (where the average is 7). $4 + 10 + 22 = \boxed{\textbf{(D) } 36}$

7. A rectangle is partitioned into 5 regions as shown. Each region is to be painted a solid color - red, orange, yellow, blue, or green - so that regions that touch are painted different colors, and colors can be used more than once. How many different colorings are possible?

   (A) 120 **(B)** 270 **(C)** 360 **(D)** 540 **(E)** 720

   Solution: We do constructive counting, starting with the bottom left block.

   - There are 5 ways to fill the bottom-left block.
   - There are now 4 ways to fill the top-left block.
   - There are now 3 ways to fill the bottom-middle block.
   - There are now 3 ways to fill the top-right block.
   - There are now 3 ways to fill the bottom-right block.

   Thus, the answer is $5 \cdot 4 \cdot 3^3 = \boxed{\textbf{(D) } 540}$

   TAKEAWAY: Casework took longer, don't be stupid

8. The infinite product
$$\sqrt[3]{10} \cdot \sqrt[3]{\sqrt[3]{10}} \cdot \sqrt[3]{\sqrt[3]{\sqrt[3]{10}}} \ldots$$
evaluates to a real number. What is that number?

(A) $\sqrt{10}$ **(B)** $\sqrt[3]{100}$ **(C)** $\sqrt[4]{1000}$ **(D)** 10 **(E)** $10\sqrt[3]{10}$

Solution: We can rewrite this as:
$$10^{\frac{1}{3}} \cdot 10^{\frac{1}{3}*\frac{1}{3}} \ldots$$
$$10^{\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots}$$
Looking at the exponent gives us an infinite geometric series with initial term $a = \frac{1}{3}$ and common ratio $r = \frac{1}{3}$. Plugging this into the sum formula gives us
$$\frac{a}{1-r} = \frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{1}{2}$$
$10^{\frac{1}{2}} = \sqrt{10} = \boxed{\textbf{(A) } \sqrt{10}}$

9. On Halloween 31 children walked into the principal's office asking for candy. They can be classified into three types: Some always lie; some always tell the truth; and some alternately lie and tell the truth. The alternaters arbitrarily choose their first response, either a lie or the truth, but each subsequent statement has the opposite truth value from its predecessor. The principal asked everyone the same three questions in this order.

”Are you a truth-teller?” The principal gave a piece of candy to each of the 22 children who answered yes.

”Are you an alternater?” The principal gave a piece of candy to each of the 15 children who answered yes.

”Are you a liar?” The principal gave a piece of candy to each of the 9 children who answered yes.

How many pieces of candy in all did the principal give to the children who always tell the truth?

(A) 7 **(B)** 12 **(C)** 21 **(D)** 27 **(E)** 31

Solution: Let T, L, A_T, A_L denote the number of children who tell the truth, lie, alternate (starting with truth), and alternate (starting with lie), respectively.
$22 = T + L + A_L$
$15 = L + A_L$ (the alternating liars are now truthers, and the alternating truthers are now liars now)
$7 = A_L$ (the alternating liars are liars again, and the alternating truthers are truthers)
Working backwards, we have that $A_L = 7, L = 8, T = 7$ The truthers only got candy once, so the answer is $\boxed{\textbf{(A) } 7}$

TAKEAWAY: Logic questions are not scary

10. What is the number of ways the numbers from 1 to 14 can be split into 7 pairs such that for each pair, the greater number is at least 2 times the smaller number?

(A) 108 **(B)** 120 **(C)** 126 **(D)** 132 **(E)** 144

Solution: Note that every number including or below 7 can be paired with numbers above it or below it. Thus, we start by considering the number 8:
$8 \implies 4$ possibilities (1,2,3,4)
$9 \implies 4 - 1 = 3$ possibilities
$10 \implies 5 - 2 = 3$ possibilities
$11 \implies 5 - 3 = 2$ possibilities
$12 \implies 6 - 4 = 2$ possibilities
$13 \implies 6 - 5 = 1$ possibility
$14 \implies 7 - 6 = 1$ possibility
Multiplying everything gives us $4 * 3 * 3 * 2 * 2 * 1 * 1 = \boxed{\textbf{(E) } 144}$

11. What is the product of all real numbers x such that the distance on the number line between $\log_6 x$ and $\log_6 9$ is twice the distance on the number line between $\log_6 10$ and 1?

(A) 10 **(B)** 18 **(C)** 25 **(D)** 36 **(E)** 81

Solution: Use logarithm formulas to rewrite this:

$$|\log_6 x = \log_6 9| = 2|\log_6 10 - \log_6 6|$$

The left side equals $\log_6 \frac{x}{9}$ or $\log_6 \frac{9}{x}$. The left side equals $2\log_6 \frac{10}{6}$ or $\log_6 \frac{6}{10}$. Evaluating the left side gives us $\log_6 \frac{25}{9}$ or $\log_6 \frac{9}{25}$.

Cross multiplying gives us that $x = 9, \frac{81}{25}$. The product equals $\boxed{\textbf{(E) } 81}$

TAKEAWAY: $n \cdot \log a = \log a^n$

12. Let $ABCD$ be a regular tetrahedron, and let M be the midpoint of $\overline{AB}$. What is $\cos(\angle CMD)$?

(A) $\frac{1}{4}$ **(B)** $\frac{1}{3}$ **(C)** $\frac{2}{5}$ **(D)** $\frac{1}{2}$ **(E)** $\frac{\sqrt{3}}{2}$

Solution: We use cosine law, and the fact that an equilateral triangle has median of $\frac{1}{\sqrt{3}}$ times its side length. Setting the side length to be 2, we want to find the larger angle in a triangle with side lengths $2, \sqrt{3}, \sqrt{3}$.

$$2^2 = 2(\sqrt{3})^2 - 2(\sqrt{3})^2(\cos\theta)$$
$$4 = 6 - 6(\cos\theta)$$
$$\cos\theta = \boxed{\textbf{(D) } \frac{1}{2}}$$

TAKEAWAY: 3D geo is usually not scary

13. Let $\mathcal{R}$ be the region in the complex plane consisting of all complex numbers z that can be written as the sum of complex numbers z_1 and z_2, where z_1 lies on the segment with endpoints 3 and $4i$, and z_2 has magnitude at most 1. What integer is closest to the area of $\mathcal{R}$?

(A) 13 **(B)** 14 **(C)** 15 **(D)** 16 **(E)** 17

Solution: We simply draw this out on the complex plane. Since we know the magnitude of $z_1 = 5$ and z_2 gives us a circle of radius 1, we end up with a 5x2 rectangle with two semicircles of radius 1 on each end. The area of this is equal to $5 \cdot 2 + \pi(1)^2 \approx 13.14 = \boxed{\textbf{(A) } 13}$

TAKEAWAY: Complex questions are not scary

14. What is the value of

$$(\log 5)^3 + (\log 20)^3 + (\log 8)(\log 0.25)$$

where log denotes the base-ten logarithm?

(A) $\frac{3}{2}$ **(B)** $\frac{7}{4}$ **(C)** 2 **(D)** $\frac{9}{4}$ **(E)** $\frac{5}{2}$

Solution: By using some sort of fucking mathematical spidey sense, we set $x = \log 2$. The reasoning for this is:

$$\log(5) = \log(10) - \log(2)$$
$$\log(20) = \log(2) + \log(10)$$
$$\log(8) = 3\log(2)$$
$$\log(\frac{1}{4}) = -2\log(2)$$

Rewriting our equation in x gives us

$$(1-x)^3 + (1+x)^2 + (3x)(-2x) = \boxed{\textbf{(C) } 2}$$

TAKEAWAY: Random numbers are usually there for a reason - use intuition

15. The roots of the polynomial $10x^3 - 39x^2 + 29x - 6$ are the height, length, and width of a rectangular box (right rectangular prism. A new rectangular box is formed by lengthening each edge of the original box by 2 units. What is the volume of the new box?

(A) $\frac{24}{5}$ **(B)** $\frac{42}{5}$ **(C)** $\frac{81}{5}$ **(D)** 30 **(E)** 48

Solution: A smarter student would have probably used Vieta's, but I just RRT and PRT bashed on the contest to find that $(x - 3)$ was a root. Dividing with synthetic division gives us

$$P(x) = (x - 3)(10x^2 - 9x + 2)$$

$$P(x) = (x - 3)(5x - 2)(2x - 1)$$

Thus, the roots are $3, \frac{2}{5}, \frac{1}{2}$. Adding 2 to each side gives us $5, \frac{12}{5}, \frac{5}{2}$. $5 \cdot \frac{12}{5} \cdot \frac{5}{2} = \boxed{\textbf{(D) } 30}$

16. A triangular number is a positive integer that can be expressed in the form $t_n = 1 + 2 + 3 + \cdots + n$, for some positive integer n. The three smallest triangular numbers that are also perfect squares are $t_1 = 1 = 1^2$, $t_8 = 36 = 6^2$, and $t_{49} = 1225 = 35^2$. What is the sum of the digits of the fourth smallest triangular number that is also a perfect square?

(A) 6 **(B)** 9 **(C)** 12 **(D)** 18 **(E)** 27

Solution:

$$1 + 2 + \ldots + n = k^2$$

$$\frac{n^2 + n}{2} = k^2$$

$$4n^2 + 4n + 1 = 8k^2 + 1$$

$$(2n + 1)^2 - 2(2k)^2 = 1$$

This is a solution to the Pell equation $a^2 - 2b^2 = 1$, the fundamental of which is $(3, 2)$. The fourth smallest can be found from $(3 + 2\sqrt{2})^4 = 577 + 408\sqrt{2}$ so $2k = 408, k = 204$ and the sum of the digits of $204^2 = 41616$ is $\boxed{\textbf{(D) } 18}$

NOTE: At this point, I don't expect you to fully solve/understand the following questions. Solutions are written out for problems that also appear on the 10A (those being 18, 19, 23, 24)

17. Suppose a is a real number such that the equation

$$a \cdot (\sin x + \sin(2x)) = \sin(3x)$$

has more than one solution in the interval $(0, \pi)$. The set of all such a can be written in the form $(p, q) \cup (q, r)$, where p, q, and r are real numbers with $p < q < r$. What is $p + q + r$?

(A) -4 **(B)** -1 **(C)** 0 **(D)** 1 **(E)** 4

Solution: By the double and triple-angle identities, $\sin(2x) = 2\sin x \cos x$ and $\sin(3x) = 3\sin x - 4\sin^3 x$. Substituting these values in, we get

$$a(\sin x + 2\sin x \cos x = 3\sin x - 4\sin x^3$$

$$a(1 + 2\cos x) = 4\cos x - 1$$

$$a = \frac{(2\cos x - 1)(2\cos x + 1)}{2\cos x + 1}$$

$$-3 < 2c - 1 < 1$$

However, this is undefined when $2\cos x + 1 = 0 \iff a = 2\cos x - 1$

Hence the solutions are $(-3, -2) \cup (-2, 1)$, which sum to $\boxed{\textbf{(A) } -4}$

18. Let T_k be the transformation of the coordinate plane that first rotates the plane k degrees counterclockwise around the origin and then reflects the plane across the y-axis. What is the least positive integer n such that performing the sequence of transformations $T_1, T_2, T_3, \ldots, T_n$ returns the point $(1,0)$ back to itself?

    (A) 359 **(B)** 360 **(C)** 719 **(D)** 720 **(E)** 721

    Solution (Q18 on AMC 10) Note that the number of degrees turned is $\frac{n\cdot(n+1)}{2}$. Thus, the smallest possible number is 359 as $359 \cdot 180 \equiv 0 \pmod{180}$. $\boxed{\textbf{(A) } 359}$

19. Suppose that 13 cards numbered $1, 2, 3, \cdots, 13$ are arranged in a row. The task is to pick them up in numerically increasing order, working repeatedly from left to right. In the example below, cards 1, 2, 3 are picked up on the first pass, 4 and 5 on the second pass, 6 on the third pass, 7, 8, 9, 10 on the fourth pass, and 11, 12, 13 on the fifth pass. For how many of the 13! possible orderings of the cards will the 13 cards be picked up in exactly two passes?

    $\boxed{7}\ \boxed{11}\ \boxed{8}\ \boxed{6}\ \boxed{4}\ \boxed{5}\ \boxed{9}\ \boxed{12}\ \boxed{1}\ \boxed{13}\ \boxed{10}\ \boxed{2}\ \boxed{3}$

    (A) 4082 **(B)** 4095 **(C)** 4096 **(D)** 8178 **(E)** 8191

    Solution: (Q22 on the AMC 10)

$$\sum_{i=0}^{13} \binom{13}{i} - 1 \implies 2^{13} - 14 \implies \boxed{\textbf{(D) } 8178}$$

20. Isosceles trapezoid $ABCD$ has parallel sides $\overline{AD}$ and $\overline{BC}$, with $BC < AD$ and $AB = CD$. There is a point P in the plane such that $PA = 1, PB = 2, PC = 3$, and $PD = 4$. What is $\frac{BC}{AD}$?

    (A) $\frac{1}{4}$ **(B)** $\frac{1}{3}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{2}{3}$ **(E)** $\frac{3}{4}$

    Solution: (Q23 on the AMC 10) Let ℓ be the vertical central axis of the trapezoid, d_1 and d_2 be half of each of the bases. Let x be the displacement of P from ℓ. Note that

$$5 = PC^2 - PB^2 = (d_1 + x)^2 - (d_1 - x)^2 = 4d_1 x$$

and

$$15 = PD^2 - PA^2 = (d_2 + x)^2 - (d_2 - x)^2 = 4d_2 x$$

hence $d_1/d_2 = \boxed{\textbf{(B) } \frac{1}{3}}$

21. Let $P(x) = x^{2022} + x^{1011} + 1$. Which of the following polynomials divides $P(x)$?

    (A) $x^2 - x + 1$ **(B)** $x^2 + x + 1$ **(C)** $x^4 + 1$ **(D)** $x^6 - x^3 + 1$ **(E)** $x^6 + x^3 + 1$

    Solution: Claim: If $x^9 = 1$, then $P(x) = 0$. Proof: We have $P(x) = x^6 + x^3 + 1$. Since x^3 is a cube root of unity, $x^6 + x^3 + 1 = 0$. $\square$

    So

$$x^9 - 1 = (x^6 + x^3 + 1)(x^3 - 1)$$

divides $P(x)$, thus $\boxed{\textbf{(E) } x^6 + x^3 + 1}$

22. Let c be a real number, and let z_1, z_2 be the two complex numbers satisfying the quadratic $z^2 - cz + 10 = 0$. Points $z_1, z_2, \frac{1}{z_1}$, and $\frac{1}{z_2}$ are the vertices of a (convex) quadrilateral Q in the complex plane. When the area of Q obtains its maximum value, c is the closest to which of the following?

    (A) 4.5 **(B)** 5 **(C)** 5.5 **(D)** 6 **(E)** 6.5

    Solution: $\boxed{\textbf{(A) } 4.5}$

23. Let h_n and k_n be the unique relatively prime positive integers such that

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}=\frac{h_n}{k_n}.$$

Let L_n denote the least common multiple of the numbers $1, 2, 3, \cdots, n$. For how many integers n with $1 \le n \le 22$ is $k_n < L_n$?

(A) 0 **(B)** 3 **(C)** 7 **(D)** 8 **(E)** 10

Solution: Bashing various primes yields $\boxed{\textbf{(D)}\ 8}$

24. How many strings of length 5 formed from the digits 0,1,2,3,4 are there such that for each $j \in \{1, 2, 3, 4\}$, at least j of the digits are less than j? (For example, 02214 satisfies the condition because it contains at least 1 digit less than 1, at least 2 digits less than 2, at least 3 digits less than 3, and at least 4 digits less than 4. The string 23404 does not satisfy the condition because it does not contain at least 2 digits less than 2.)

(A) 500 **(B)** 625 **(C)** 1089 **(D)** 1199 **(E)** 1296

Solution (Q24 on the AMC 10): Consider strings of length 5 formed from digits 0 to 5. Now, write the numbers in a row, but whenever a number is repeated, increase it by 1 until it is no longer repeated, taking numbers modulo 6. Then, the condition is equivalent to writing the numbers 0 to 4 in some order. Exactly $\frac{1}{6}$ of the strings satisfy this, so the number of working strings is $\frac{1}{6} \cdot 6^5 = \boxed{\textbf{(E)}\ 1296}$

25. A circle with integer radius r is centered at (r, r). Distinct line segments of length c_i connect points $(0, a_i)$ to $(b_i, 0)$ for $1 \le i \le 14$ and are tangent to the circle, where a_i, b_i, and c_i are all positive integers and $c_1 \le c_2 \le \cdots \le c_{14}$. What is the ratio $\frac{c_{14}}{c_1}$ for the least possible value of r?

(A) $\frac{21}{5}$ **(B)** $\frac{85}{13}$ **(C)** 7 **(D)** $\frac{39}{5}$ **(E)** 17

Solution: $\boxed{\textbf{(E)}\ 17}$